## Photometry of rotating regular n-sided prisms for arbitrary solar phase angles

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### Abstract

The photometric signal from a rotating regular n-sided prism is analyzed theoretically for arbitrary solar phase angles. All $N$ faces of the prism are assumed to be diffuse reflectors with the same reflectance. The analysis shows that the range of the periodic signal (peak minus trough) varies with phase angle, and contains a finite number of peaks. The separation between the peaks is inversely related to the number of faces of the prism.

### 1. Introduction

A theoretical analysis [1] was previously completed of the periodic photometric signal returned from a spinning satellite with a cross-section in the shape of a regular polygon. The analysis was restricted to a zero degree solar phase angle, i.e. the illumination source is located behind the observer. In this paper, a general solution for the photometric signal is derived for the case of an arbitrary phase angle, with any value between $0$ and $\pi$. A simulation was performed for arbitrary solar phase angles and the theoretical and analytical results are in complete agreement.

### 2. Diffuse Model

Each face of the rotating prism is assumed to be a flat diffuse surface of constant reflectance. All faces are assumed to be identical in size, shape and reflectance. The cross-section of the rotating object is a regular polygon with $N$ sides. The normal to each face intersects the axis of rotation and is separated from the normal of the neighboring faces by an angle of $2\pi/N$ radians in either direction. As a result, when the object has undergone a rotation of $2\pi/N$ radians, the new state cannot be distinguished from the previous state.

The solar phase angle is the angle between the Sun vector and the observer vector. It is shown in Figure 1 by the symbol, $\Omega$. As is implied in this figure, this paper will restrict its analysis to the case where the axis of rotation is perpendicular to the plane containing the Sun and the observer.

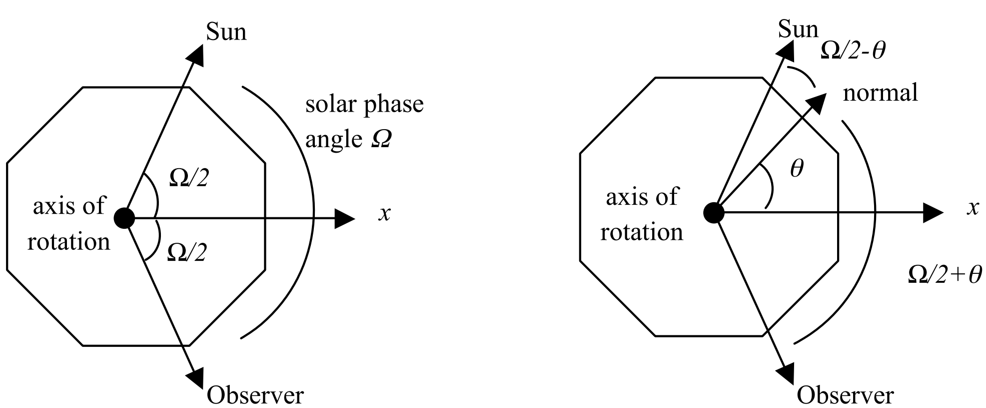


Fig. 1. The solar phase angle is the angle between the vectors from the object to the Sun and to the observer. If face, $k$, is at an angle $\theta$ to the x-axis, then the angles between the normal and the Sun and the observer are $\Omega/2\pm\theta$.

The amount of light seen by the observer, is the sum of the light from all the exposed faces. This quantity is proportional to the sum of the product of the reflectance of each face with the cosine of the angle between each normal and the Sun vector and the cosine of the angle between each normal and the observer vector. Using the angles defined in Figure 1, the photometric signature from the light received by the observer is proportional to:

$$p(t) = aA \sum_{k=-N/2}^{N/2-1} \cos(\Omega/2 + \theta_k)_+ \cos(\Omega/2 - \theta_k)_+ \tag{1}$$

where the angles $\theta_k$ and $\Omega$, although not shown explicitly, are functions of time. The constant, $a$, is the albedo or the reflectance of each surface and the constant, $A$, is the true area of each surface. Since all faces are assumed to be the same reflectance, the constant $a$ can be set to unity with no loss in generality.

The plus sign subscript notation on the cosine functions indicates that the cosine function contributes to the signal only when the cosine value is positive, otherwise it has a zero contribution. The first cosine function will be negative when that face is out of view of the observer, while the second cosine function will be negative when that face is in shadow and is not illuminated.

We will assume that at time $t=0$, there is one face whose normal is pointing along the x-axis, as shown in Fig. 1. The $N$ faces will be numbered $-N/2$ to $N/2-1$, where the $0^{th}$ face is the one whose normal points along the x-axis. Since there are $N$ faces around $2\pi$ radians, the normals for adjacent faces are separated by a fixed angle of $2\pi/N$. With a constant rotation rate of $\omega$ radians/second, the angle of the normal of the $k^{th}$ face is given by:

$$\theta_k(t) = \omega t + k\frac{2\pi}{N}, \qquad \text{for } -\frac{N}{2} \le k \le \frac{N}{2} - 1 \tag{2}$$

The photometric signal from the $k^{th}$ face at time $t$ is therefore given by:

$$p_k(t) = A\cos\left(\Omega/2 + \omega t + k\frac{2\pi}{N}\right)_+ \cos\left(\Omega/2 - \omega t - k\frac{2\pi}{N}\right)_+ \tag{3}$$

The total photometric signal is the sum of the signals from the individual faces:

$$p(t) = \sum_{k=-N/2}^{N/2-1} A\cos\left(\Omega/2 + \omega t + k\frac{2\pi}{N}\right)_+ \cos\left(\Omega/2 - \omega t - k\frac{2\pi}{N}\right)_+ \tag{4}$$

This equation can be further evaluated by substituting the width of one face of the prism, $2R\sin(\pi/N)$ from reference [1], where $R$ is the radius of the encircling cylinder, and multiplying by the height of the cylinder, $H$. We will also substitute the symbol, $\alpha$, for the rotation angle, $\omega t$.

$$p(t) = 2RH\sin\left(\frac{\pi}{N}\right) \sum_{k=-N/2}^{N/2-1} \cos\left(\Omega/2 + \alpha + k\frac{2\pi}{N}\right)_+ \cos\left(\Omega/2 - \alpha - k\frac{2\pi}{N}\right)_+ \tag{5}$$

$$\text{where} \quad 0 \le \alpha < \frac{2\pi}{N} \quad \text{over one period of the rotation cycle.} \tag{6}$$

## 3. Summation Limits

The plus sign subscripts on the cosine functions in the summation in Eq. 5 can be removed by limiting $k$ to values where the cosine functions are positive, or in other words, when their arguments are smaller than $\pi/2$. This condition is expressed by the two inequalities:

$$\begin{aligned} &\Omega/2 + \alpha + k\frac{2\pi}{N} \le \frac{\pi}{2} \\ &\Omega/2 - \alpha - k\frac{2\pi}{N} \le \frac{\pi}{2} \end{aligned} \tag{7}$$

Solving for $k$ in the above inequalities yields the following limits for $k$:

$$-\frac{N}{4}\left(1-\frac{\Omega-2\alpha}{\pi}\right) \le k \le \frac{N}{4}\left(1-\frac{\Omega+2\alpha}{\pi}\right) \tag{8}$$

Since $k$ must be an integer, the limits of $k$ can be expressed as the integer part of these values, i.e.

$$-\,Int\left(\frac{N}{4}\left(1-\frac{\Omega-2\alpha}{\pi}\right)\right) \le k \le Int\left(\frac{N}{4}\left(1-\frac{\Omega+2\alpha}{\pi}\right)\right) \tag{9}$$

Using these limits, the equation for the photometric signal becomes:

$$p(t) = 2RH\sin\left(\frac{\pi}{N}\right)\sum_{k=k_1}^{k_2}\cos\left(\Omega/2+\alpha+k\frac{2\pi}{N}\right)\cos\left(\Omega/2-\alpha-k\frac{2\pi}{N}\right) \tag{10}$$

$$\text{where} \quad k_1 = -Int\left(\frac{N}{4}\left(1-\frac{\Omega-2\alpha}{\pi}\right)\right) \quad \textit{and} \quad k_2 = Int\left(\frac{N}{4}\left(1-\frac{\Omega+2\alpha}{\pi}\right)\right) \tag{11}$$

### *3.1 Upper Limit*

Expanding the upper limit, $k_2$, in Eq. 11, yields:

$$k_2 = Int\left(\frac{N}{4}-\frac{N\Omega}{4\pi}-\frac{\alpha}{2\pi/N}\right) \tag{12}$$

The terms inside the integer function can be expressed as the sum of an integer part and a fractional part, e.g.,

$$\frac{N}{4} = Int\left(\frac{N}{4}\right) + frac\left(\frac{N}{4}\right) \tag{13}$$

where the fractional part is greater than or equal to zero, but is less than unity, i.e.

$$0 \le frac\left(\frac{N}{4}\right) < 1 \tag{14}$$

Using this formalism, the upper limit becomes:

$$k_2 = Int\left( Int\left(\frac{N}{4}\right) + frac\left(\frac{N}{4}\right) - Int\left(\frac{N\Omega}{4\pi}\right) - frac\left(\frac{N\Omega}{4\pi}\right) - \frac{\alpha}{2\pi/N} \right) \tag{15}$$

Note from Eq. 6, that the last term containing $\alpha$, by definition, only has a fractional part. The integer parts within the integer function can be removed yielding:

$$k_2 = Int\left(\frac{N}{4}\right) - Int\left(\frac{N\Omega}{4\pi}\right) + Int\left( frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) - \frac{\alpha}{2\pi/N} \right) \tag{16}$$

This works as long as the sum of the fractional parts within the integer function in Eq. 16 is positive. When this sum is negative, it pulls one of the integer parts below the threshold level. Since these integer parts were already removed from the integer function, this effect can be accounted for by properly defining the behavior of the integer function for negative arguments, while not changing the operation for positive arguments. For example,

$$Int(f) = \begin{cases} -1 & for \quad -1 \le f < 0 \\ 0 & for \quad 0 \le f < 1 \end{cases} \tag{17}$$

The rationale for Eq. 17 is that a slight negative fractional sum will pull the integer parts below the integer threshold, while a slight positive fraction will not push the number into the next integer. To simplify the notation, the integer term, $I$, and fractional term, $f$, are defined by:

$$k_2 = I + Int(f) \tag{18}$$

where

$$I = Int\left(\frac{N}{4}\right) - Int\left(\frac{N\Omega}{4\pi}\right) \tag{19}$$

and

$$f = frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) - \frac{\alpha}{2\pi/N} \tag{20}$$

The first term in Eq. 20, *frac(N/4)*, can only have one of four values, *0, ¼, ½, and ¾*, while the other two terms vary continuously between 0 and 1. From Eq. 20, therefore:

$$-2 + frac\left(\frac{N}{4}\right) < f \le frac\left(\frac{N}{4}\right) \tag{21}$$

and therefore the integer part of $f$ varies between:

$$-2 \le Int(f) \le 0 \tag{22}$$

This means that *Int(f)* has the values, *-2*, *-1* and *0*, when the following inequalities hold:

$$Int(f) = \begin{cases} -2 & -2 + frac\left(\frac{N}{4}\right) < frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) - \frac{\alpha}{2\pi/N} < -1 \\ -1 & -1 \le frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) - \frac{\alpha}{2\pi/N} < 0 \\ 0 & 0 \le frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) - \frac{\alpha}{2\pi/N} \le frac\left(\frac{N}{4}\right) \end{cases} \tag{23}$$

This results in six inequalities that restrict $\alpha$. By solving for $\alpha$ in each of the six inequalities, the following is true:

$$Int(f) = \begin{cases} -2 & \begin{cases} \frac{\alpha}{2\pi/N} < 2 - frac\left(\frac{N\Omega}{4\pi}\right) & \}\text{always true} \\ \frac{\alpha}{2\pi/N} > 1 + frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) & \}\text{when } frac\left(\frac{N}{4}\right) < frac\left(\frac{N\Omega}{4\pi}\right) < 1 \end{cases} \\ -1 & \begin{cases} \frac{\alpha}{2\pi/N} \le 1 + frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) & \}\text{when } frac\left(\frac{N}{4}\right) < frac\left(\frac{N\Omega}{4\pi}\right) < 1 \\ \frac{\alpha}{2\pi/N} > frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) & \}\text{when } 0 \le frac\left(\frac{N\Omega}{4\pi}\right) \le frac\left(\frac{N}{4}\right) \end{cases} \\ 0 & \begin{cases} \frac{\alpha}{2\pi/N} \le frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) & \}\text{when } 0 \le frac\left(\frac{N\Omega}{4\pi}\right) \le frac\left(\frac{N}{4}\right) \\ \frac{\alpha}{2\pi/N} \ge -frac\left(\frac{N\Omega}{4\pi}\right) & \}\text{always true} \end{cases} \end{cases} \tag{24}$$

These are expressed graphically in Figure 2, showing the values that *Int(f)* takes in different regions of α, depending on the relationship of the two fractions in Eq. 24.

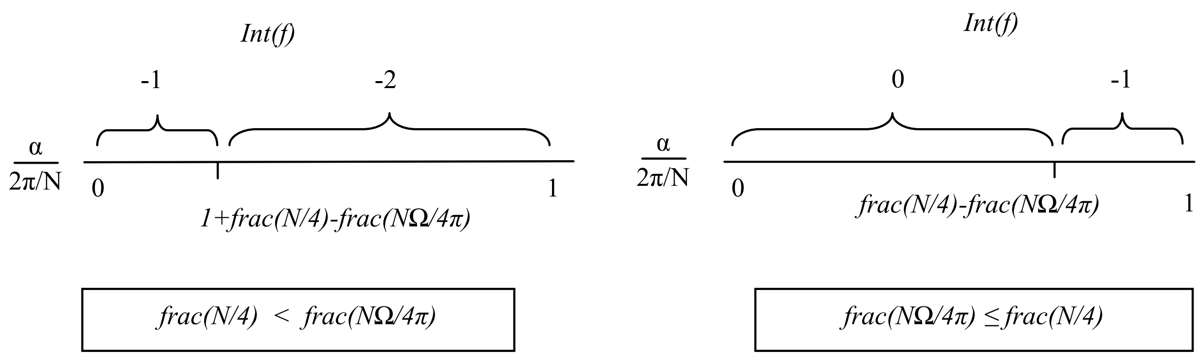


Figure 2. The integer part of the fraction takes on specific values depending on the value of $\alpha$ and the two fractions.

This graph shows that the upper limit will be the integer part, *I*, from Eq. 19 minus a small number depending on the rotation angle and the value of the solar phase angle.

*3.2 Lower Limit*

Similarly, the lower limit, $k_1$, in Eq. 11, can be expanded:

$$k_1 = -Int\left(\frac{N}{4} - \frac{N\Omega}{4\pi} + \frac{\alpha}{2\pi/N}\right) \tag{25}$$

In a similar manner, the limit can be separated into integer parts.

$$k_1 = -Int\left(\frac{N}{4}\right) + Int\left(\frac{N\Omega}{4\pi}\right) - Int\left(frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) + \frac{\alpha}{2\pi/N}\right) \tag{26}$$

or

$$k_1 = -I - Int(f) \tag{27}$$

where

$$I = Int\left(\frac{N}{4}\right) - Int\left(\frac{N\Omega}{4\pi}\right) \tag{28}$$

and

$$f = frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) + \frac{\alpha}{2\pi/N} \tag{29}$$

Note that this is not the same expression as in Equations 18 through 20.

Since the fractions in Eq. 29 vary between 0 and just less than 1, the fraction, *f*, is restricted to the range:

$$-1 + frac\left(\frac{N}{4}\right) < f < 1 + frac\left(\frac{N}{4}\right) \tag{30}$$

and therefore the integer part of *f* varies between:

$$-1 \le Int(f) \le 1 \tag{31}$$

This means that *Int(f)* has the values, *-1*, *0* and *1*, when the following inequalities hold:

$$Int(f) = \begin{cases} -1 & -1 + frac\left(\frac{N}{4}\right) < frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) + \frac{\alpha}{2\pi/N} < 0 \\ 0 & 0 \le frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) + \frac{\alpha}{2\pi/N} < 1 \\ 1 & 1 \le frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right) + \frac{\alpha}{2\pi/N} < 1 + frac\left(\frac{N}{4}\right) \end{cases} \tag{32}$$

This results in six inequalities that restrict $\alpha$. By solving for $\alpha$ in each of the six inequalities, the following is true:

$$
Int(f) = \begin{cases}
-1 & \begin{cases} \dfrac{\alpha}{2\pi/N} > -1 + frac\left(\dfrac{N\Omega}{4\pi}\right) & \} \text{always true} \\[2ex] \dfrac{\alpha}{2\pi/N} < frac\left(\dfrac{N\Omega}{4\pi}\right) - frac\left(\dfrac{N}{4}\right) & \} \text{when } frac\left(\dfrac{N}{4}\right) < frac\left(\dfrac{N\Omega}{4\pi}\right) < 1 \end{cases} \\[6ex]
0 & \begin{cases} \dfrac{\alpha}{2\pi/N} \geq frac\left(\dfrac{N\Omega}{4\pi}\right) - frac\left(\dfrac{N}{4}\right) & \} \text{when } frac\left(\dfrac{N}{4}\right) \leq frac\left(\dfrac{N\Omega}{4\pi}\right) < 1 \\[2ex] \dfrac{\alpha}{2\pi/N} < 1 + frac\left(\dfrac{N\Omega}{4\pi}\right) - frac\left(\dfrac{N}{4}\right) & \} \text{when } 0 \leq frac\left(\dfrac{N\Omega}{4\pi}\right) \leq frac\left(\dfrac{N}{4}\right) \end{cases} \\[6ex]
1 & \begin{cases} \dfrac{\alpha}{2\pi/N} \geq 1 + frac\left(\dfrac{N\Omega}{4\pi}\right) - frac\left(\dfrac{N}{4}\right) & \} \text{when } 0 \leq frac\left(\dfrac{N\Omega}{4\pi}\right) < frac\left(\dfrac{N}{4}\right) \\[2ex] \dfrac{\alpha}{2\pi/N} < 1 + frac\left(\dfrac{N\Omega}{4\pi}\right) & \} \text{always true} \end{cases}
\end{cases} \tag{33}
$$

These are expressed graphically in Figure 3, showing the values that *Int(f)* takes in different regions of α, depending on the relationship of the two fractions in Eq. 33.

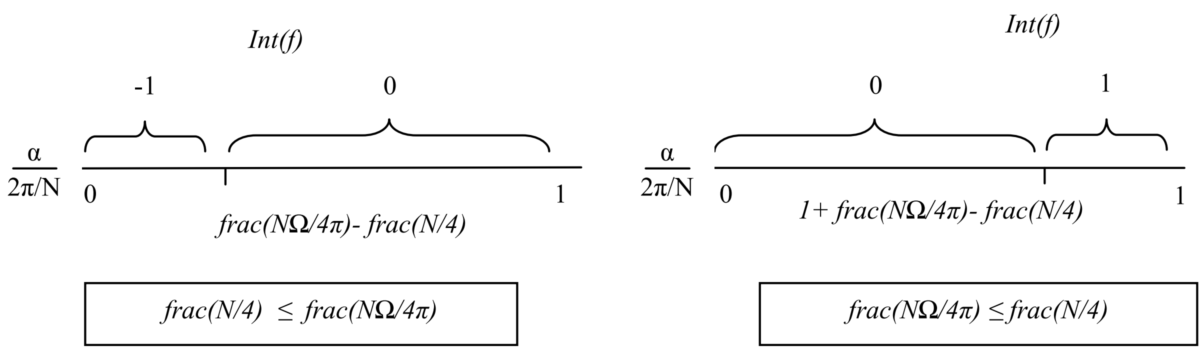


Figure 3. The integer part of the fraction takes on specific values depending on the value of $\alpha$ and the two fractions.

This graph shows that the upper limit will be the integer part, *I*, from Eq. 28 plus a small number depending on the rotation angle and the value of the solar phase angle.

*3.3 Combined Limits*

Note to equate the upper and lower limit fraction ranges, the upper limit will be adjusted by -1. Therefore, the summations will be defined to range from limits $k_1$ to $k_2$, where:

$$k_1 = -I - A \tag{34}$$

and

$$k_2 = I - 1 + B \tag{35}$$

The constant, *A*, is shown in Figure 3 and *B-1* is shown in Figure 2. The values of *A* and *B* depend on the relationship of the two fractions and the value of $\alpha$.

If we define:

$$f = abs\left( frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right)\right) \tag{36}$$

then, in Figure 4, we can see that when $frac(N/4) < frac(N\Omega/4\pi)$, there are two threshold values creating three regions of $\alpha$ where the limits are constant. The two threshold values are $f$ and $1\text{-}f$. For the two cases of $f \le ½$ and $f \ge ½$ the constants $A$ and $B$ have the following values:

$$\{A \quad B\} = \left\{ \begin{bmatrix} -1 & 0 & \text{for } 0 \le \frac{\alpha}{2\pi/N} < f \\ 0 & 0 & \text{for } f \le \frac{\alpha}{2\pi/N} < 1-f \\ 0 & -1 & \text{for } 1-f \le \frac{\alpha}{2\pi/N} < 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & \text{for } 0 \le \frac{\alpha}{2\pi/N} < 1-f \\ -1 & -1 & \text{for } 1-f \le \frac{\alpha}{2\pi/N} < f \\ 0 & -1 & \text{for } f \le \frac{\alpha}{2\pi/N} < 1 \end{bmatrix} \right\} \tag{37}$$

where the left half is for $f \le ½$ and the right half is for $f \ge ½$.

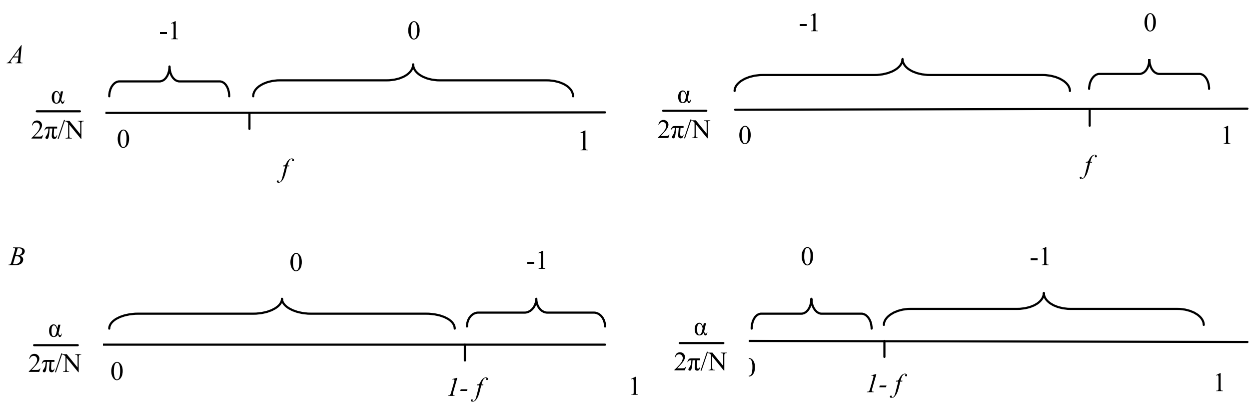


Figure 4. The values of $A$ and $B$, when $frac(N/4) < frac(N\Omega/4\pi)$, for $f \le ½$ (left) and $f \ge ½$ (right) .

A similar result occurs for the case of $frac(N\Omega/4\pi) < frac(N/4)$. In this case, the right halves of Figures 2 and 3 describe the thresholds. The same threshold defined in Equation 36 is used and shown in Figure 5.

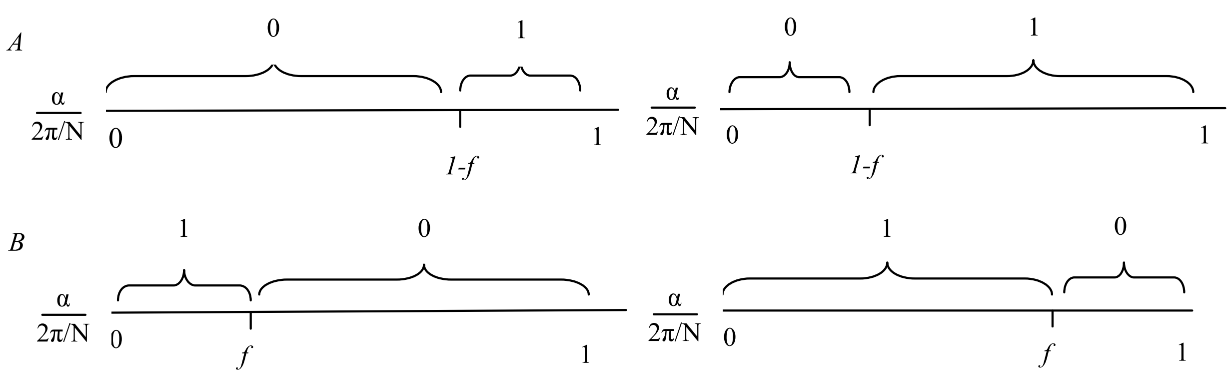


Figure 5. The values of $A$ and $B$, when $frac(N\Omega/4\pi) < frac(N/4)$, for $f < ½$ (left) and $f \ge ½$ (right) .

The two constants $A$ and $B$ have the following values:

$$\{A \quad B\} = \left\{ \begin{array}{ccl} 0 & 1 & \text{for } 0 \le \dfrac{\alpha}{2\pi/N} \le f \\ 0 & 0 & \text{for } f < \dfrac{\alpha}{2\pi/N} < 1-f \\ 1 & 0 & \text{for } 1-f \le \dfrac{\alpha}{2\pi/N} < 1 \end{array} \right\} \left\{ \begin{array}{ccl} 0 & 1 & \text{for } 0 \le \dfrac{\alpha}{2\pi/N} < 1-f \\ 1 & 1 & \text{for } 1-f \le \dfrac{\alpha}{2\pi/N} \le f \\ 1 & 0 & \text{for } f < \dfrac{\alpha}{2\pi/N} < 1 \end{array} \right\} \tag{38}$$

where the left half is for $f < ½$ and the right half is for $f \geq ½$. The equation for the photometric signal becomes:

$$p(t) = 2RH \sin\left(\frac{\pi}{N}\right) \sum_{k=k_1}^{k_2} \cos\left(\Omega/2 + \alpha + k\frac{2\pi}{N}\right) \cos\left(\Omega/2 - \alpha - k\frac{2\pi}{N}\right) \tag{39}$$

$$\text{where} \quad k_1 = -I - A, \;\; k_2 = I - 1 + B \tag{40}$$

## 4. Evaluation of the Summation

The cosine functions with the summation now can be further simplified using the following relationship:

$$\cos(\phi + \theta)\cos(\phi - \theta) = \frac{1}{2}\cos(2\phi) + \frac{1}{2}\cos(2\theta) \tag{41}$$

This relationship is easy to show by expanding the product of the two cosines and combining terms. Using this relationship, the photometric signal becomes:

$$p(t) = RH \sin\left(\frac{\pi}{N}\right) \sum_{k=k_1}^{k_2} \cos(\Omega) + RH \sin\left(\frac{\pi}{N}\right) \sum_{k=k_1}^{k_2} \cos\left(2\alpha + k\frac{4\pi}{N}\right) \tag{42}$$

where the limits of the summations are defined in Eq. 40 and the values of $A$ and $B$ are summarized in Table 1.

| *case* | *A* | *B* | *α/2π/N* | *f* | *frac(N/4)<frac(NΩ/4π)* |
|---|---|---|---|---|---|
| 2 | -1 | 0 | *0 ≤ α < f* | *f ≤ ½* | |
| 1 | 0 | 0 | *f ≤ α < 1-f* | *f ≤ ½* | |
| 3 | 0 | -1 | *1-f ≤ α < 1* | *f ≤ ½* | |
| | | | | | |
| 2 | -1 | 0 | *0 ≤ α <1-f* | *f ≥ ½* | |
| 4 | -1 | -1 | *1-f ≤ α < f* | *f ≥ ½* | |
| 3 | 0 | -1 | *f ≤ α < 1* | *f ≥ ½* | |
| | | | | | *frac(NΩ/4π)≤frac(N/4)* |
| 6 | 0 | 1 | *0 ≤ α ≤ f* | *f < ½* | |
| 1 | 0 | 0 | *f < α < 1-f* | *f < ½* | |
| 5 | 1 | 0 | *1-f ≤ α < 1* | *f < ½* | |
| | | | | | |
| 6 | 0 | 1 | *0 ≤ α <1-f* | *f ≥ ½* | |
| 7 | 1 | 1 | *1-f ≤ α ≤ f* | *f ≥ ½* | |
| 5 | 1 | 0 | *f < α < 1* | *f ≥ ½* | |

Table 1. The values of $A$ and $B$, for the 7 different cases.

Ignoring the constant in front, the photometric signal in Equation 42 consists of two summations, one over a constant and one over a variable. The first summation is:

$$\sum_{k=k_1}^{k_2} \cos(\Omega) = \cos(\Omega)[k_2 - k_1 + 1] \tag{43}$$

or

$$\sum_{k=k_1}^{k_2} \cos(\Omega) = \cos(\Omega)[2I + A + B] \tag{44}$$

The second summation can be expanded using the expansion for the cosine of the sum of two angles.

$$\sum_{k=k_1}^{k_2} \cos\left(2\alpha + k\frac{4\pi}{N}\right) = \cos(2\alpha)\sum_{k=k_1}^{k_2} \cos\left(k\frac{4\pi}{N}\right) - \sin(2a)\sum_{k=k_1}^{k_2} \sin\left(k\frac{4\pi}{N}\right) \tag{45}$$

The summations on the right-hand side of Eq. 45 can be further reduced by substituting the limits from Eq. 40. The summation over the sine function is:

$$\sum_{k=k_1}^{k_2} \sin(k\theta) = \sum_{k=-I-A}^{I-1+B} \sin(k\theta) \tag{46}$$

or

$$\sum_{k=k_1}^{k_2} \sin(k\theta) = \sum_{k=-I}^{I-1} \sin(k\theta) - A\sin\left(\left(I + \frac{(A+1)}{2}\right)\theta\right) + B\sin\left(\left(I + \frac{(B-1)}{2}\right)\theta\right) \tag{47}$$

This formulation works, because the constants *A* and *B* have the values -1, 0 or 1. Since all but one of the sine terms, in the summation in Equation 47, cancel with their negative counterparts, this equation becomes:

$$\sum_{k=k_1}^{k_2} \sin(k\theta) = -\sin(I\theta) - A\sin\left(\left(I + \frac{(A+1)}{2}\right)\theta\right) + B\sin\left(\left(I + \frac{(B-1)}{2}\right)\theta\right) \tag{48}$$

Similarly, the summation over the cosine function is:

$$\sum_{k=k_1}^{k_2} \cos(k\theta) = \sum_{k=-I-A}^{I-1+B} \cos(k\theta) \tag{49}$$

or

$$\sum_{k=k_1}^{k_2} \cos(k\theta) = \sum_{k=-I}^{I-1} \cos(k\theta) + A\cos\left(\left(I + \frac{(A+1)}{2}\right)\theta\right) + B\cos\left(\left(I + \frac{(B-1)}{2}\right)\theta\right) \tag{50}$$

or

$$\sum_{k=k_1}^{k_2} \cos(k\theta) = 2\sum_{k=0}^{I-1} \cos(k\theta) + \cos(I\theta) - 1 + A\cos\left(\left(I + \frac{(A+1)}{2}\right)\theta\right) + B\cos\left(\left(I + \frac{(B-1)}{2}\right)\theta\right) \tag{51}$$

The one-sided summation over the cosine function has the following solution [2, 3]:

$$\sum_{k=0}^{I-1}\cos(k\theta)=\frac{\sin\left(\frac{I\theta}{2}\right)\cos\left(\frac{(I-1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \tag{52}$$

Substituting Eq. 52 into Eq. 51 is:

$$\sum_{k=k_1}^{k_2}\cos(k\theta)=2\frac{\sin\left(\frac{I\theta}{2}\right)\cos\left(\frac{(I-1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}+\cos(I\theta)-1+A\cos\left(\left(I+\frac{(A+1)}{2}\right)\theta\right)+B\cos\left(\left(I+\frac{(B-1)}{2}\right)\theta\right) \tag{53}$$

Expanding the cosine function and combining the terms yields:

$$\sum_{k=k_1}^{k_2}\cos(k\theta)=2\frac{\sin(I\theta)\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}+A\cos\left(\left(I+\frac{(A+1)}{2}\right)\theta\right)+B\cos\left(\left(I+\frac{(B-1)}{2}\right)\theta\right) \tag{54}$$

Substituting Equations 54 and 48 into Equation 45 and combining all the terms yields:

$$\sum_{k=-k_1}^{k_2}\cos(2\alpha+k\theta)=\frac{\cos\left(2\alpha-\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\sin(I\theta)+A\cos\left(2\alpha-\left(I+\frac{(A+1)}{2}\right)\theta\right)$$

$$+B\cos\left(2\alpha+\left(I+\frac{(B-1)}{2}\right)\theta\right) \tag{55}$$

Substituting $\theta = 4\pi/N$ and using that $I = Int(N/4)-Int(N\Omega/4\pi)$ gives:

$$\sum_{k=-k_1}^{k_2}\cos\left(2\alpha+k\frac{4\pi}{N}\right)=\frac{\cos\left(2\alpha-\frac{2\pi}{N}\right)}{\sin\left(\frac{2\pi}{N}\right)}\sin\left(I\frac{4\pi}{N}\right)+A\cos\left(2\alpha-\left(I+\frac{(A+1)}{2}\right)\frac{4\pi}{N}\right)$$

$$+B\cos\left(2\alpha+\left(I+\frac{(B-1)}{2}\right)\frac{4\pi}{N}\right) \tag{56}$$

Together with Equation 56, the photometric signal from Equation 42 becomes:

$$p(t)=RH\sin\left(\frac{\pi}{N}\right)\left[\cos(\Omega)(2I+A+B)+\sum_{k=-k_1}^{k_2}\cos\left(2\alpha+k\frac{4\pi}{N}\right)\right] \tag{57}$$

From Table 1, it can be seen that there are 7 cases to consider. For each of the seven cases, the values of A and B are substituted into Equations 56 and 57 and the common terms are expanded and re-combined. When this is done, the functions that result are listed in Table 2, below. The seven individual cases are indicated with the appropriate subscript.

| A | B | Function |
|---|---|---|
| 0 | 0 | $p_1(t) = RH\sin(\pi/N)\left[2I\cos(\Omega) + \frac{\cos(2\alpha - 2\pi/N)}{\sin(2\pi/N)}\sin\left(I\frac{4\pi}{N}\right)\right]$ |
| -1 | 0 | $p_2(t) = RH\sin(\pi/N)\left[(2I-1)\cos(\Omega) + \frac{\cos(2\alpha)}{\sin(2\pi/N)}\sin\left(\left(I - \frac{1}{2}\right)\frac{4\pi}{N}\right)\right]$ |
| 0 | -1 | $p_3(t) = RH\sin(\pi/N)\left[(2I-1)\cos(\Omega) + \frac{\cos(2\alpha - 4\pi/N)}{\sin(2\pi/N)}\sin\left(\left(I - \frac{1}{2}\right)\frac{4\pi}{N}\right)\right]$ |
| -1 | -1 | $p_4(t) = RH\sin(\pi/N)\left[(2I-2)\cos(\Omega) + \frac{\cos(2\alpha - 2\pi/N)}{\sin(2\pi/N)}\sin\left((I-1)\frac{4\pi}{N}\right)\right]$ |
| 1 | 0 | $p_5(t) = RH\sin(\pi/N)\left[(2I+1)\cos(\Omega) + \frac{\cos(2\alpha - 4\pi/N)}{\sin(2\pi/N)}\sin\left(\left(I + \frac{1}{2}\right)\frac{4\pi}{N}\right)\right]$ |
| 0 | 1 | $p_6(t) = RH\sin(\pi/N)\left[(2I+1)\cos(\Omega) + \frac{\cos(2\alpha)}{\sin(2\pi/N)}\sin\left(\left(I + \frac{1}{2}\right)\frac{4\pi}{N}\right)\right]$ |
| 1 | 1 | $p_7(t) = RH\sin(\pi/N)\left[(2I+2)\cos(\Omega) + \frac{\cos(2\alpha - 2\pi/N)}{\sin(2\pi/N)}\sin\left((I+1)\frac{4\pi}{N}\right)\right]$ |

Table 2. The functions for each of the 7 different sets of values of *A* and *B*.

From Table 2 above, it can be seen that all of the functions are of the form of *cos(2α)*. Some of the cases are of this form, but shifted by an amount equal to *2π/N*, the length of the periodicity of the signal. Due to the symmetry of the cosine function around zero, the shifted functions are just the mirror image of the unshifted functions.

## 5. Regions of Validity

The regions of validity of these different cases are derived from the results in Table 1. It can be seen that the functions group in threes, depending on the relative values of the two fractions in Eq. 36. In addition, those groups of three functions are valid in three regions of *α*, also defined by the same relationship. The values of the functions are dependent on the integer parameter, *I*, defined in Eq. 19. For clarity, those equations are repeated here:

$$I = Int\left(\frac{N}{4}\right) - Int\left(\frac{N\Omega}{4\pi}\right) \qquad (19)$$

$$f = abs\left(frac\left(\frac{N}{4}\right) - frac\left(\frac{N\Omega}{4\pi}\right)\right) \qquad (36)$$

To better understand these relationships, we will define the number of faces, *N*, in the following manner:

$$N = 4Q + q \qquad (58)$$

where Q and q are both integers. Clearly, they are related to the integer and fractional parts of *N/4* by:

$$Int\left(\frac{N}{4}\right) = Q \tag{59}$$

and

$$frac\left(\frac{N}{4}\right) = {}^{q}\!/_{4} \tag{60}$$

These integer and fractional parts only depend on the number of faces of the prism. The other integer and fraction from Eqs. 19 and 36 depend directly on the value of the phase angle, **Ω**.

To study the fractional part, *frac(NΩ/4π)*, a schematic plot is shown in Figure 6. The fraction will be zero at $\Omega=0$ and will increase linearly until the fractional argument equals unity. At that point, the phase angle has the value:

$$\Omega = \frac{4\pi}{N} \tag{61}$$

The value of the fraction at that location drops back to zero. This behavior repeats, as **Ω** increases, dropping back to zero at multiple points where the argument of the fraction is a multiple of the value in Eq. 61. These phase angles are:

$$\Omega = k\frac{4\pi}{N} \tag{62}$$

where $k$ is an integer. This happens $Q$ times, until the last drop occurs at $k=Q$, where the phase angle is:

$$\Omega = Q\frac{4\pi}{N} = \pi\left(\frac{Q}{Q+q/4}\right) = \pi\left(\frac{Q+q/4-q/4}{Q+q/4}\right) = \pi\left(1-q/N\right) \tag{63}$$

The last region continues to increase linearly until $\Omega=\pi$, where

$$frac\left(\frac{N\Omega}{4\pi}\right) = frac\left(\frac{N}{4}\right) = q/4 \tag{64}$$

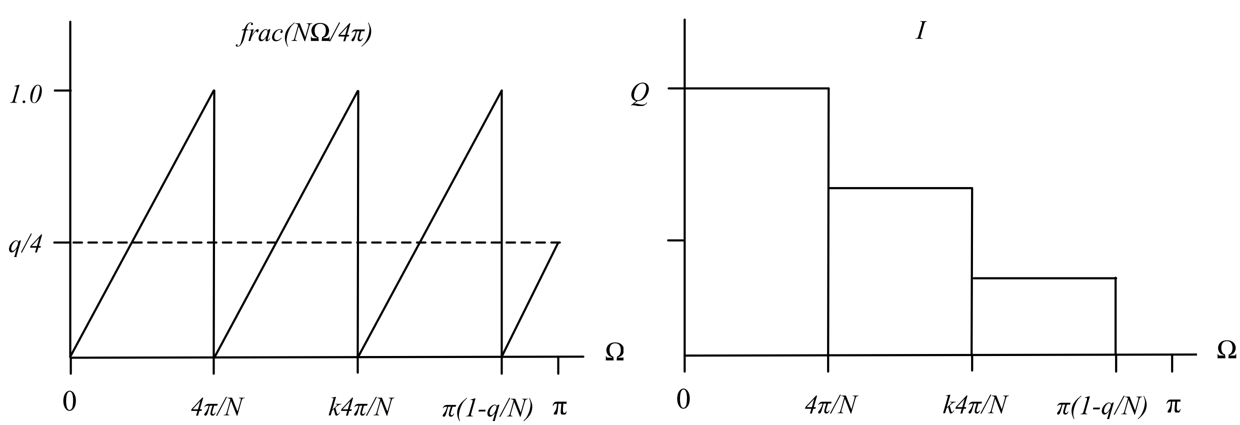


Figure 6. The values of *A* and *B*, when *frac(NΩ/4π) < frac(N/4)*, for *f* < ½ (left) and *f* ≥ ½ (right) .

The integer value, $I$, is also plotted in Figure 6. From Eq. 19, it can be seen that at $\Omega=0$, the value of $Int(N\Omega/4\pi)=0$ and therefore the value of $I$ is:

$$I = Int\left(\frac{N}{4}\right) - Int\left(\frac{N\Omega}{4\pi}\right) = Q \tag{65}$$

The value of $I$ remains constant until $\Omega=4\pi/N$, where the value of $Int(N\Omega/4\pi)=1$ and the integer, $I$, drops to $Q-1$. This happens several times at each multiple of $4\pi/N$. The value of $I$ finally drops to zero at $\Omega=\pi(1-q/N)$.

In terms of these alternate variables, $I$ can be written as:

$$I = Int\left(\frac{N}{4}\right) - Int\left(\frac{N\Omega}{4\pi}\right) = Q - k \tag{66}$$

$$\text{where} \quad 0 \le k \le Q$$

The variable $k$ is just the index of the $Q+1$ regions starting with 0 at the left and ending with $Q$ on the right. This formulation of the integer, $I$, can be substituted into the function forms listed in Table 2. The integer, $I$, occurs in the form of $sin((I+a)4\pi/N)$, where $a = 0, \pm½, \pm1$. Substituting Eq. 66 into the sine function yields:

$$\sin\left((I+a)\frac{4\pi}{N}\right) = \sin\left((Q-k+a)\frac{4\pi}{N}\right) = \sin\left(\pi\left((Q+q/4-q/4-k+a)\frac{4\pi}{N}\right)\right) \tag{67}$$

$$= \sin\left(\pi\left(\frac{Q+q/4}{N/4} - \frac{(k+q/4-a)}{N/4}\right)\right) = \sin\left(\pi - (k+q/4-a)\frac{4\pi}{N}\right) = \sin\left((k+q/4-a)\frac{4\pi}{N}\right) \tag{68}$$

In other words, the quantity, $I+a$, is replaced by $k+q/4-a$. With this change, the 7 functions become:

| A | B | Function |
|---|---|---|
| 0 | 0 | $p_1(t) = RH\sin(\pi/N)\left[2I\cos(\Omega) + \frac{\cos(2\alpha - 2\pi/N)}{\sin(2\pi/N)}\sin\left((k+q/4)\frac{4\pi}{N}\right)\right]$ |
| -1 | 0 | $p_2(t) = RH\sin(\pi/N)\left[(2I-1)\cos(\Omega) + \frac{\cos(2\alpha)}{\sin(2\pi/N)}\sin\left((k+q/4+½)\frac{4\pi}{N}\right)\right]$ |
| 0 | -1 | $p_3(t) = RH\sin(\pi/N)\left[(2I-1)\cos(\Omega) + \frac{\cos(2\alpha - 4\pi/N)}{\sin(2\pi/N)}\sin\left((k+q/4+½)\frac{4\pi}{N}\right)\right]$ |
| -1 | -1 | $p_4(t) = RH\sin(\pi/N)\left[(2I-2)\cos(\Omega) + \frac{\cos(2\alpha - 2\pi/N)}{\sin(2\pi/N)}\sin\left((k+q/4+1)\frac{4\pi}{N}\right)\right]$ |
| 1 | 0 | $p_5(t) = RH\sin(\pi/N)\left[(2I+1)\cos(\Omega) + \frac{\cos(2\alpha - 4\pi/N)}{\sin(2\pi/N)}\sin\left((k+q/4-½)\frac{4\pi}{N}\right)\right]$ |
| 0 | 1 | $p_6(t) = RH\sin(\pi/N)\left[(2I+1)\cos(\Omega) + \frac{\cos(2\alpha)}{\sin(2\pi/N)}\sin\left((k+q/4-½)\frac{4\pi}{N}\right)\right]$ |
| 1 | 1 | $p_7(t) = RH\sin(\pi/N)\left[(2I+2)\cos(\Omega) + \frac{\cos(2\alpha - 2\pi/N)}{\sin(2\pi/N)}\sin\left((k+q/4-1)\frac{4\pi}{N}\right)\right]$ |

Table 3. The functions for each of the 7 different sets of values of $A$ and $B$.

The ranges of solar phase angle, Ω, for which these functions are valid are defined in Table 1. The ranges of *α* for which the functions are valid depend on the fractional parameter, *f*, that is defined in Eq. 36. An expanded version of Figure 6 is shown in Figs. 7 and 8, where the regions of phase angle where the functions are valid are shown.

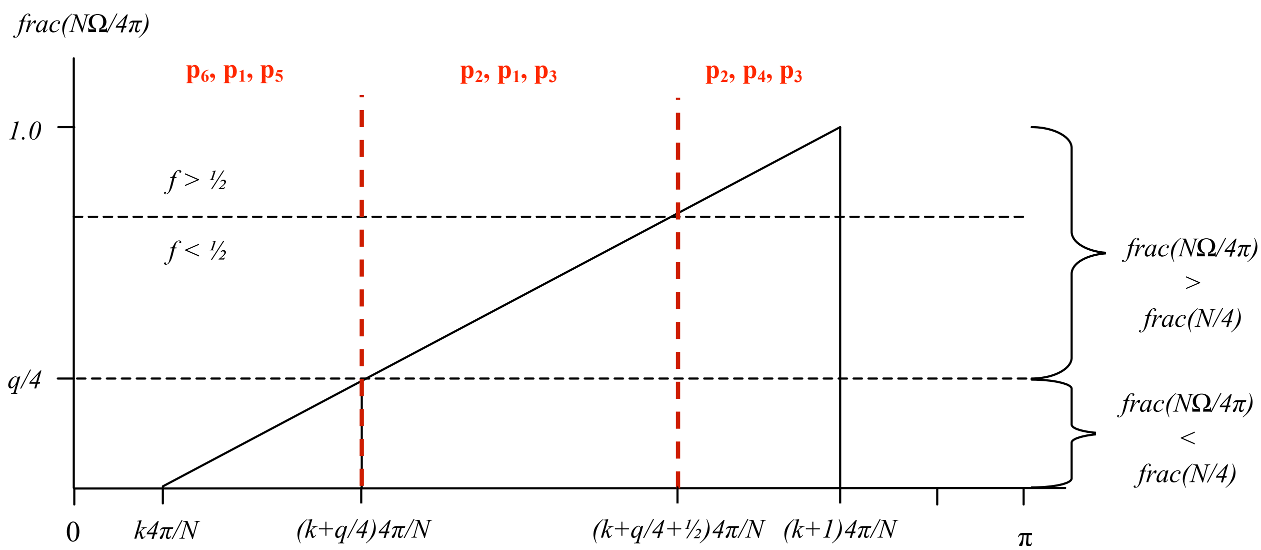


Figure 7. The regions of validity when $q < ½$.

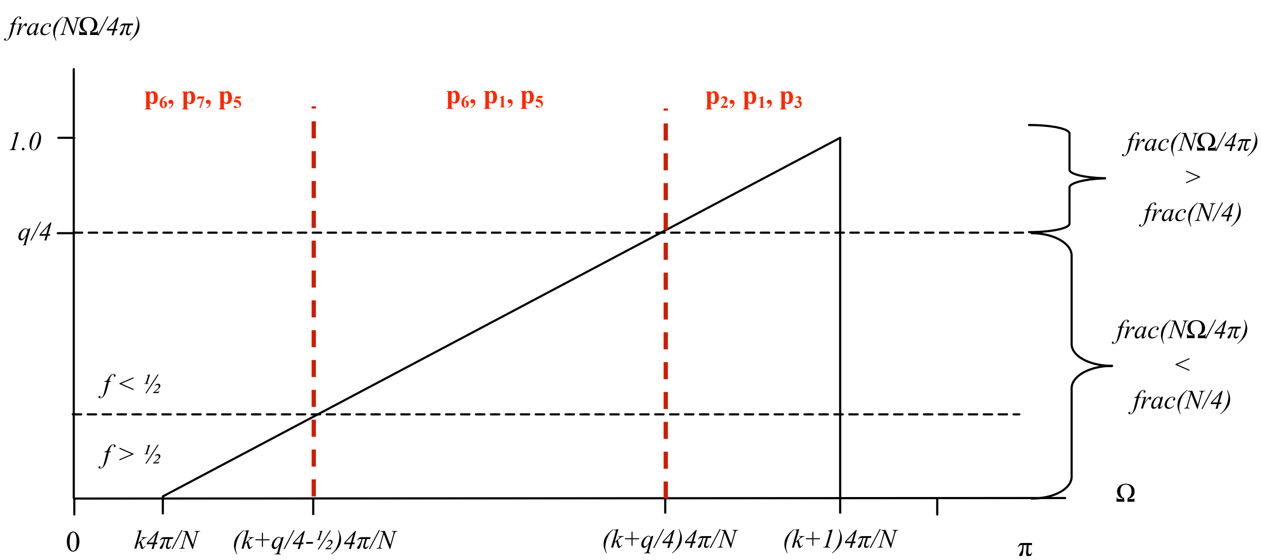


Figure 8. The regions of validity when $q > ½$.

The two figures above, Figs. 7 and 8, show four regions, which are delineated by whether the difference of *frac(N/4)-frac(NΩ/4π)* is positive or negative and by whether the absolute value of the difference is less than or greater than ½.

For an even number of faces, when $q=0$ and $q=2$, only two regions can occur as the phase angle changes. For an odd number of faces (the cases shown above), when $q=1$ and $q=3$, then three of the four regions are valid as the phase angle changes.

## 6. Simulation

A simulation was performed to compare the numerical evaluation of Equation 5 with the theoretical results from Tables 1 and 3. The phase angle was varied in 100 equal increments from $0$ to $\pi$. For each value of phase angle, the rotation angle was varied in 1,000 equal increments of the rotation angle, $\alpha$, from $0$ to $2\pi/N$.

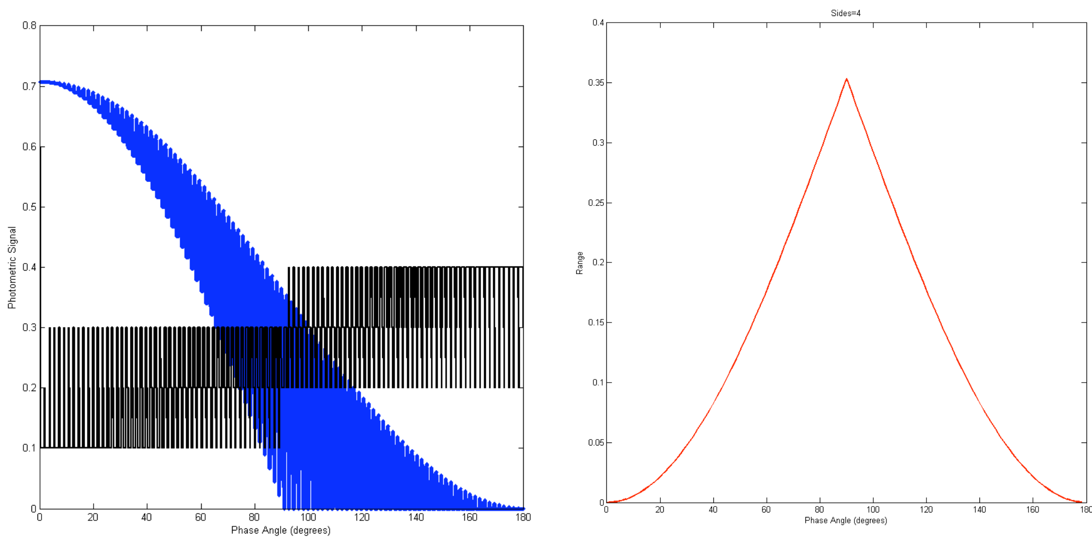


Figure 9. Theoretical result versus simulation. The simulation for $N=4$ is plotted, on the left, in red lines and the theoretical result with blue points. The two agree so well, that only one is visible. On the right is the range, i.e. peak-to-trough value, as a function of phase angle.

The code to generate the following figures is given in Figure 13. The case of $N=4$ is shown in Figure 9. The blue line is the theoretical result from Table 3. The horizontal axis is the phase angle expressed in degrees. Each cycle visible in this figure is a single cycle of the rotation angle, $\alpha$, for a different value of the phase angle. It is clear that as the phase angle increases, the photometric signal decreases, until it goes to zero at a phase angle of $\pi$.

It is also apparent that the range of the signal (the difference of the maximum and minimum values over one cycle) changes with phase angle. It is small for small phase angles and also for large phase angles. It is maximum at 90 degrees. The range is plotted on the right side of Figure 9.

The black lines, on the left in Figure 9, indicate which theoretical function was used to plot the blue points. On the left of the figure, the black lines vary between 2, 1 and 3, showing that the ($p_2$, $p_1$, $p_3$) triplet was used to calculate the values. On the right side, the ($p_2$, $p_4$, $p_3$) triplet was used.

For this case, $N=4$, the integer and fractional parts of $N/4$ are $Q=1$ and $q=0$. That means that all of the left part of Figure 9 corresponds to one region in the phase angle space, specifically $k=0$. From Figure 7, when $q=0$ these two triplets are the only ones needed to describe the function across all values of the phase angle.

In Figure 10, the case of $N=5$ is shown. For this case, $Q=1$ and $q=1$. The occurrence of triplets, as shown on the left in Figure 10, corresponds to the arrangement shown in Figure 7. In the range plot, on the right side of Figure 10, there are two peaks. By comparing the two plots in Figure 10, it can be seen that the maxima occur at the junctions where the triplets change. These locations are also where $f=0$ and $f=½$. From Figure 7, the angular separation between the locations of $f=0$ and $f=½$ is equal to $2\pi/N$.

Figure 11 and 12 show the results for cases $N=6$ and $N=7$. For both of these cases, $Q=1$, but $q=2$ and $q=3$, respectively. The arrangement of the triplets agrees with the diagrams in Figs. 7 and 8. Also, the locations of the peaks correspond to the locations where $f=0$ and $f=½$. This distance agrees with the angular separation of $2\pi/N$, independent of the value of $N$. As the number of faces increases, the separation between the peaks decreases.

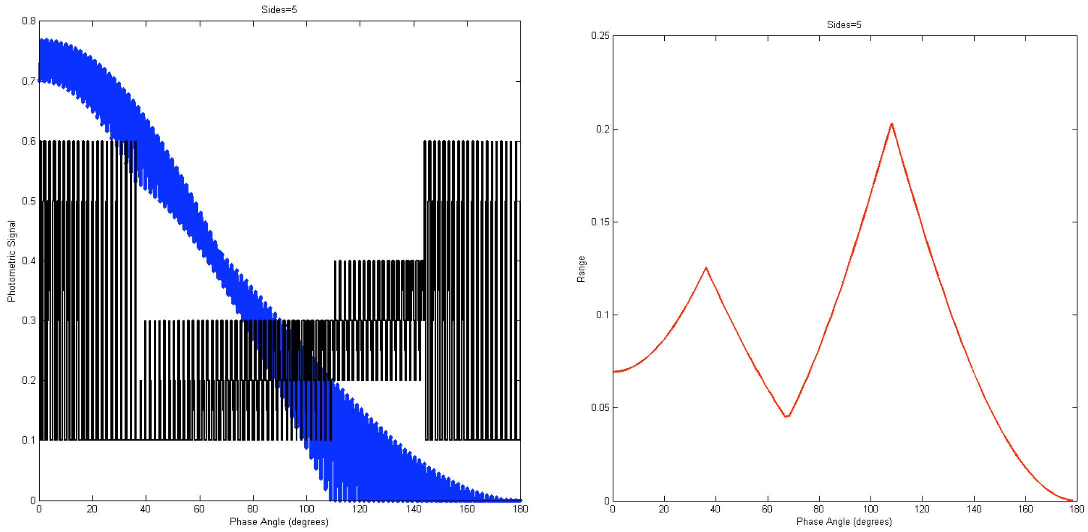


Figure 10.  Theoretical result versus simulation for N=5.

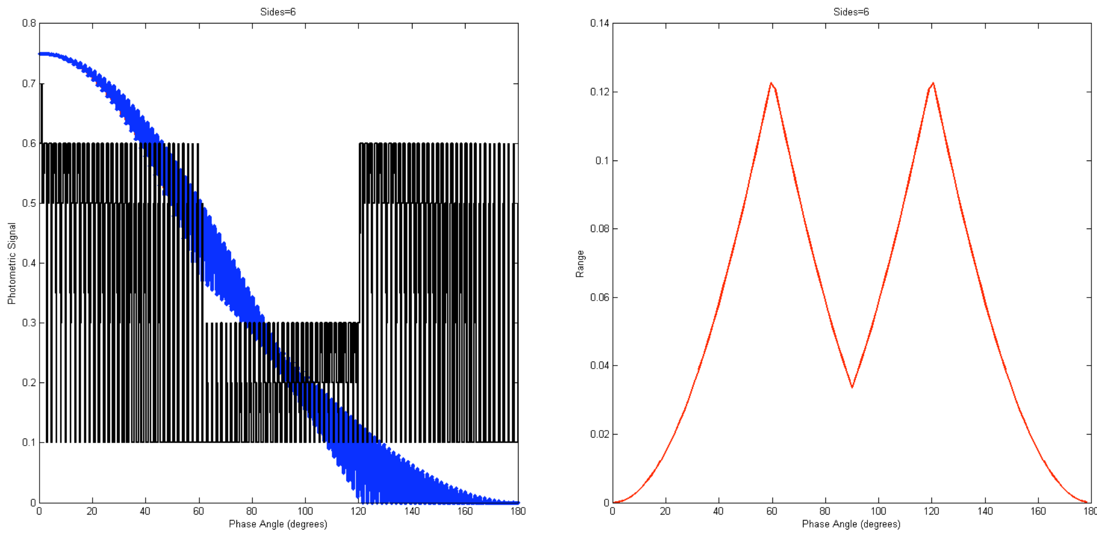


Figure 11.  Theoretical result versus simulation for N=6.

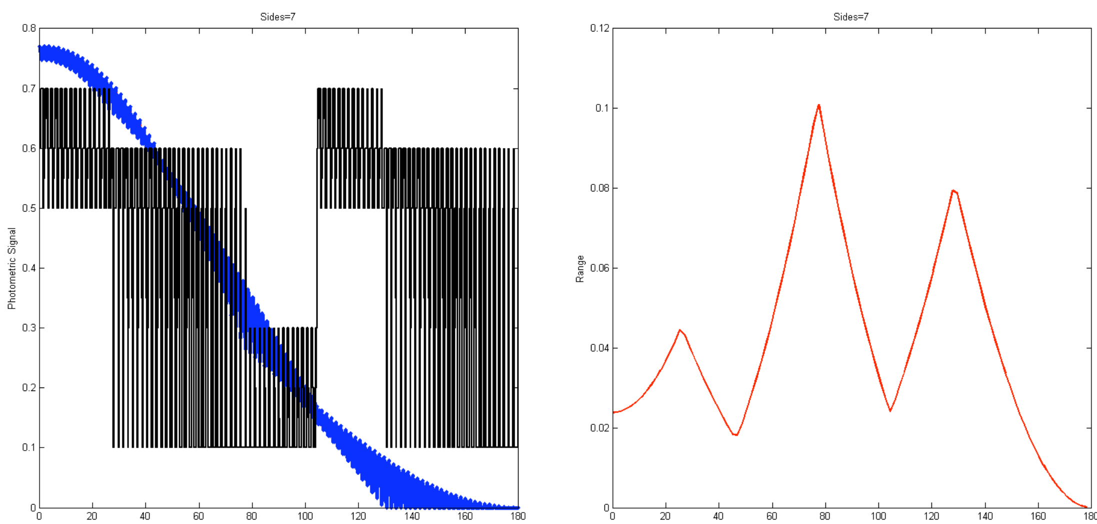


Figure 12.  Theoretical result versus simulation for N=7.

## 7. Simulation Code

```
% spinning_prism.m
% This script plots simulation vs. theory for a spinning prism

cycle = zeros(1,100000);
range = zeros(11,100);
left = zeros(1,1000);
right = zeros(1,1000);

for N=4:11

    Q = floor(N/4);
    q = N-4*Q;
    frac_N = q/4;

for m=0:99
    % Single facet
    phase_angle = m*(pi)/100;
    albedo_area = sin(pi/N);
    angle_facet = 2*pi/N;

    % Do 1000 samples per cycle
    theta = repmat(2*pi*[0:(N-1)]'/N,[1 1000])+repmat([0:999]*2*pi/(1000*N),[N 1]);
    observer = cos(phase_angle/2+theta);
    sun = cos(phase_angle/2-theta);
    diffuse = sum(albedo_area*observer.*(observer > 0).*sun.*(sun > 0));
    cycle(1000*m+1:1000*(m+1)) = diffuse;

    % Calculate the range
    maxv = max(diffuse(:));
    minv = min(diffuse(:));
    range(N,m+1) = (maxv-minv);
    real_max(m+1) = max(diffuse(:));
    real_min(m+1) = min(diffuse(:));

    % Theoretical Curves
    k = floor((N*phase_angle)/(4*pi));
    frac_NOmega = (N*phase_angle)/(4*pi)-k;
    f = abs(frac_N-frac_NOmega);
    I = Q-k;
    P1 = 0.5*sin(pi/N)*(2*I*cos(phase_angle)+cos(2*[0:999]*2*pi/(1000*N)-2*pi/N)*sin(I*4*pi/N)/sin(2*pi/N));
    P2 = 0.5*sin(pi/N)*((2*I-1)*cos(phase_angle)+cos(2*[0:999]*2*pi/(1000*N))*sin((I-0.5)*4*pi/N)/sin(2*pi/N));
    P3 = 0.5*sin(pi/N)*((2*I-1)*cos(phase_angle)+cos(2*[0:999]*2*pi/(1000*N)-4*pi/N)*sin((I-
0.5)*4*pi/N)/sin(2*pi/N));
    P4 = 0.5*sin(pi/N)*((2*I-2)*cos(phase_angle)+cos(2*[0:999]*2*pi/(1000*N)-2*pi/N)*sin((I-
1)*4*pi/N)/sin(2*pi/N));
    P5 = 0.5*sin(pi/N)*((2*I+1)*cos(phase_angle)+cos(2*[0:999]*2*pi/(1000*N)-
4*pi/N)*sin((I+0.5)*4*pi/N)/sin(2*pi/N));
    P6 = 0.5*sin(pi/N)*((2*I+1)*cos(phase_angle)+cos(2*[0:999]*2*pi/(1000*N))*sin((I+0.5)*4*pi/N)/sin(2*pi/N));
    P7 = 0.5*sin(pi/N)*((2*I+2)*cos(phase_angle)+cos(2*[0:999]*2*pi/(1000*N)-
2*pi/N)*sin((I+1)*4*pi/N)/sin(2*pi/N));
```

```
% Pick the proper functions
if frac_N < frac_NOmega
        if f < 0.5
            left = ((0:999)/1000) < f;
            right = ((0:999)/1000) >= 1-f;
            theory(1000*m+1:1000*(m+1)) = P2.*left+P1.*(1-left).*(1-right)+P3.*right;
            number(1000*m+1:1000*(m+1)) = (0.2).*left+(0.1).*(1-left).*(1-right)+(+0.3).*right;
        else
            left = ((0:999)/1000) < 1-f;
            right = ((0:999)/1000) >= f;
            theory(1000*m+1:1000*(m+1)) = P2.*left+P4.*(1-left).*(1-right)+P3.*right;
            number(1000*m+1:1000*(m+1)) = (0.2).*left+(0.4).*(1-left).*(1-right)+(+0.3).*right;
        end
    else
        if f < 0.5
            left = ((0:999)/1000) <= f;
            right = ((0:999)/1000) >= 1-f;
            theory(1000*m+1:1000*(m+1)) = P6.*left+P1.*(1-left).*(1-right)+P5.*right;
            number(1000*m+1:1000*(m+1)) = (0.6).*left+(0.1).*(1-left).*(1-right)+(+0.5).*right;
        else
            left = ((0:999)/1000) < 1-f;
            right = ((0:999)/1000) > f;
            theory(1000*m+1:1000*(m+1)) = P6.*left+P7.*(1-left).*(1-right)+P5.*right;
            number(1000*m+1:1000*(m+1)) = (0.6).*left+(0.7).*(1-left).*(1-right)+(+0.5).*right;
        end
    end
end

% The theory function has exceedingly small negative values
theory = theory.*(theory > 0);

% Plot the result
figure;
plot(1.8*((1:100000)-1)/1000,cycle,'r',1.8*((1:100000)-1)/1000,theory,'b.',1.8*((1:100000)-
1)/1000,number,'k','linewidth',2);
title(sprintf('Sides=%g',N));
xlabel('Phase Angle (degrees)');
ylabel('Photometric Signal');
axis square;

figure;
plot(1.8*((1:100)-1),range(N,:),'r','linewidth',2);
title(sprintf('Sides=%g',N));
xlabel('Phase Angle (degrees)');
ylabel('Range');
axis square;

end
```

Figure 10.  Matlab script to generate figures comparing the simulation with the theoretical results.

## 8. Conclusions

A spinning n-sided prism has been shown to produce a photometric signal that is periodic. Given that all faces of the regular prism have the same reflectance, the angular period of the signal is equal to $2\pi/N$, i.e. it is $N$ times smaller than the true rotation period.

The range of the signal (peak-to-trough value) has been shown to vary significantly with phase angle. There is an overall decrease in the range of the signal as the phase angle increases. Although not shown here, it can be shown that as $N$ goes to infinity, in the limit, the photometric signal is identical to that of a cylinder.

The range of the signal has also been shown to vary periodically with phase angle. The period of this variation, i.e. the distance between peaks, is equal to $2\pi/N$. If it is possible to detect 2 or more peaks in the range of the signal, then it should be possible to extract the number of faces of the prism.

In a similar manner, one can determine how much of a change in phase angle is needed to detect the number of faces. For example, if the prism has 8 faces, then the angular separation between peaks in the range will be $\pi/4$, or 45 degrees. To be certain of detecting multiple peaks for an eight-sided prism, therefore, one should measure the photometry over a change of phase angle of at least 90 degrees.

This result is theoretical and should be used to better understand the nature of the photometry of a spinning prism. The theory has yet to be tested on actual data, so as yet, no conclusions can be drawn as to the practicality of the method for determining the number of faces on a spinning, regular n-sided prism.

## 9. References


1. Knox, K. T., "Resolving Rotational Ambiguities for Spin-stabilized Satellites", 2008 AMOS Technical Conference, pp. 366-379, (2008).
2. L. B. W. Jolley, Summation of Series, Dover Publications, Inc., New York, NY, p. 78 (1961).
3. The solution given in reference [2] for Equation 196 is wrong. The correct solution was determined from Mathematica and confirmed by recursion.