Orbit Determination and Maneuver Detection Using Event Representation with Thrust-Fourier-Coefficients

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ABSTRACT

The classical orbit determination (OD) method of dealing with unknown maneuvers is to restart the OD process with post-maneuver observations. However, it is also possible to continue the OD process through such unknown maneuvers by representing those unknown maneuvers with an appropriate event representation. Event representation using Thrust-Fourier-Coefficients (TFCs) rigorously provides a unique control law that can generate an equivalent orbital change for a given unknown maneuver. This paper presents applications of this representation approach to real-time orbit tracking problem across unknown maneuvers. The modified extended Kalman filter with TFCs is capable of fitting tracking data in real time and maintaining an OD solution in the presence of unknown maneuvers. Simulation results also show that the modified filter is found effective in detecting a sudden change in TFC values which indicates a maneuver.

1. INTRODUCTION

The space catalog has been growing dramatically in recent years and maintaining tracking of space objects is essential to enhance space situational awareness (SSA) capability in Earth orbit. However, tracking of a maneuvering satellite is a difficult task especially when a maneuver is related to an unknown perturbing event. Any unknown change of force model is regarded as an unknown maneuver, which can be caused by unplanned maneuver, structural deployment, explosion, collision with space debris, or some drastic changes in space environment [1]. Without compensating for unknown perturbations acting on orbiting satellites, incoming data after an unknown maneuver will be rejected and a satellite will be lost.

A number of different algorithms have been developed to track a maneuvering satellite. The most common approach is to use the conventional Kalman filter to estimate the state of a satellite, but in the presence of an unknown maneuver, its performance is seriously degraded. To account for an unknown maneuver, various techniques have been introduced and they can be classified into three categories: equivalent noise, input detection and estimation, and switching model approach [2]. The equivalent noise approach assumes that unknown dynamics can be sufficiently modeled by a white or colored process noise [3]. It estimates satellite states in the presence of these unknown noises, and noise compensation (SNC) and dynamic model compensation (DMC) are included in this approach [4]. It appears that this approach is relatively simple but encounters a difficult problem of finding equivalent noise level and related noise statistics in real time.

The input detection and estimation approach is to detect a maneuver and to estimate an unknown acceleration input explicitly from the available measurement data [2]. Using the estimated control input, the satellite state is estimated directly without having a maneuver model. Main examples of this approach are the input estimation (IE) technique [5], the modified input estimation (MIE) technique [6] and the enhanced Input Estimation (EIE) technique [7]. These algorithms apply some simplifying assumptions that require unknown maneuvers to be constant acceleration input, to have a certain maneuver duration, or to have a known maneuver onset time. These assumptions are quite restrictive in real satellite tracking applications under unknown maneuvers. If these assumptions made do not correspond to the actual nature of the unknown maneuver, filter performance may be degraded.

The switching model approach describes the motion of a satellite with a non-maneuver model and a maneuver model [2]. Upon receiving measurement data, the algorithm decides which model to use in real time. The variable state dimension (VSD) approach [8], the generalized pseudo-Bayesian (GPB) approach [9], the interacting multiple model (IMM) algorithm [10], and the adaptive IMM (AIMM) algorithm [11] can be included in this approach. The performance of this approach heavily depends on good maneuver detection, but often it is difficult to detect maneuver termination time. Also, it will be challenging to apply this approach to a low-thrust maneuver case due to a delay of maneuver detection and a difficulty of finding an appropriate maneuver model.

The tracking algorithm presented in this paper falls into the input detection and estimation approach. It distinguishes itself from other algorithms in that it does not rely on any assumption about a maneuver. It utilizes an event representation technique with thrust-Fourier-coefficients (TFCs) to explicitly estimate an unknown control acceleration. Previous work by Ko and Scheeres has shown that any maneuver performed by a satellite transitioning between two arbitrary orbital states can be represented as an equivalent maneuver connecting those two states using TFCs [12]. Applying this concept, a follow-up study by Ko and Scheeres has developed an unknown maneuver detection scheme by modifying the sequential filter with 14 TFCs as a post-maneuver analysis [13]. In this paper, an extended Kalman filter (EKF) is modified with TFCs in order to track a maneuvering satellite in real time. This algorithm estimates the state and the control input jointly without requiring any maneuver assumption or a priori information of a maneuver.

This paper begins with brief explanation of how to represent an unknown maneuver with 14 TFCs in section II. The subsequent subsection describes how to maintain tracking of a satellite by modifying the EKF with event representation method using 14 TFCs. In section III, the performance of the proposed EKF algorithm is verified by applying the filter to 3 different types of unknown maneuvers: an impulsive, a 30-minute continuous burn, or a low-thrust maneuver. Concluding remarks are given in the final section.

2. TRACKING A MANEUVERING SATELLITE WITH MODIFIED EXTENDED KALMAN FILTER USING THRUST-FOURIER-COEFFICIENTS

In this section, a mathematical model for unknown dynamics representation using TFCs is briefly reviewed, and a modified extended Kalman filter is presented to enable us to maintain tracking of a maneuvering satellite.

2.1. Event Representation with TFCs

When a spacecraft undergoes an unknown maneuver, the motion of the spacecraft can be described with the perturbing thrust acceleration arising linearly in its dynamics equations [14]:

$$\dot{\vec{X}} = \vec{F}(\vec{X}, t) + B \cdot \vec{a}_u(t) \tag{1}$$

$$\frac{d}{dt}\begin{bmatrix}\vec{r}(t)\\ \dot{\vec{r}}(t)\end{bmatrix}_{6\times 1} = \begin{bmatrix}\dot{\vec{r}}(t)\\ \ddot{\vec{r}}(t)\end{bmatrix}_{6\times 1} + \begin{bmatrix}0_{3\times 3}\\ I_{3\times 3}\end{bmatrix}_{6\times 3} \cdot \vec{a}_u(t)\,_{3\times 1} \tag{2}$$

where $\vec{X}$ is the satellite position and velocity state, $\vec{F}(\vec{X}, t)$ is the a priori known dynamics, and $\vec{a}_u(t)$ is the true perturbing acceleration for an unknown maneuver. Representing the perturbing acceleration as a Fourier series expansion in the eccentric anomaly (E), the acceleration component in each direction (R : radial, S : circumferential, and W : normal) can be expressed in terms of thrust Fourier coefficients (α_k, β_k)[15] :

$$\vec{a}_u(t) = U_R(t)\hat{\boldsymbol{r}} + U_S(t)\hat{\boldsymbol{s}} + U_W(t)\hat{\boldsymbol{w}} \tag{3}$$

$$\begin{aligned} U_R(t) &= \sum_{k=0}^{\infty}\left[\alpha_k^R \cos kE(t) + \beta_k^R \sin kE(t)\right] \\ U_S(t) &= \sum_{k=0}^{\infty}\left[\alpha_k^S \cos kE(t) + \beta_k^S \sin kE(t)\right] \\ U_W(t) &= \sum_{k=0}^{\infty}\left[\alpha_k^W \cos kE(t) + \beta_k^W \sin kE(t)\right] \end{aligned} \tag{4}$$

In previous work by Hudson and Scheeres, the equations of motion were averaged over the eccentric anomaly, and the averaged dynamics equations were found to be a function of 14 TFCs, $\vec{C}$ [16]:

$$\vec{C} = [\alpha_0^R, \alpha_1^R, \alpha_2^R, \beta_1^R, \alpha_0^S, \alpha_1^S, \alpha_2^S, \beta_1^S, \beta_2^S, \alpha_0^W, \alpha_1^W, \alpha_2^W, \beta_1^W, \beta_2^W]^T \tag{5}$$

For a given orbital change across an unknown maneuver, there are an infinite number of ways to connect the pre-maneuver orbit state with the post-maneuver orbit state using different control laws. Ko and Scheeres found a simple way to obtain a unique solution for the 14 TFCs set analytically that can effectively represent an unknown perturbing acceleration [17]:

$$\vec{C} = \overline{G}(\vec{œ})^T \cdot \left[\overline{G}(\vec{œ}) \cdot \overline{G}(\vec{œ})^T\right]^{-1} \cdot \left[\frac{\vec{œ}(t_f) - \vec{œ}(t_0)}{t_f - t_0}\right] \tag{6}$$

where $\overline{G}(\vec{œ})$ is a matrix form of the averaged Gauss equations while $\vec{œ}(t_f)$ and $\vec{œ}(t_0)$ are the post-maneuver state and the pre-maneuver state respectively. It computes 14 TFCs that control the secular motion of a satellite across an unknown maneuver to reach the given post-maneuver state at a given time. With the computed 14 TFCs from Eq. (6), the control profile for an unknown maneuver can be represented as follows :

$$\begin{aligned} \vec{a}_u \simeq\, & (\alpha_0^R + \alpha_1^R \cos E + \beta_1^R \sin E + \alpha_2^R \cos 2E)\hat{\boldsymbol{r}} \\ & + (\alpha_0^S + \alpha_1^S \cos E + \beta_1^S \sin E + \alpha_2^S \cos 2E + \beta_2^S \sin 2E)\hat{\boldsymbol{s}} \qquad (7) \\ & + (\alpha_0^W + \alpha_1^W \cos E + \beta_1^W \sin E + \alpha_2^W \cos 2E + \beta_2^W \sin 2E)\hat{\boldsymbol{w}} \\ & = \begin{bmatrix}\hat{\boldsymbol{r}} & \hat{\boldsymbol{s}} & \hat{\boldsymbol{w}}\end{bmatrix} \begin{bmatrix} A1 & zeros & zeros \\ zeros & A2 & zeros \\ zeros & zeros & A3 \end{bmatrix} \vec{C} \qquad (8) \\ & = T \cdot S \cdot \vec{C} \qquad (9) \\ A1 =\, & \begin{bmatrix}1 & \cos E & \cos 2E & \sin E\end{bmatrix} \\ A2 = A3 =\, & \begin{bmatrix}1 & \cos E & \cos 2E & \sin E & \sin 2E\end{bmatrix} \end{aligned}$$

in which T is the transformation matrix ($\hat{\boldsymbol{r}} = \frac{\vec{r}}{|\vec{r}|}$, $\hat{\boldsymbol{w}} = \frac{\vec{r}\times\vec{v}}{|\vec{r}\times\vec{v}|}$, $\hat{\boldsymbol{s}} = \hat{\boldsymbol{w}} \times \hat{\boldsymbol{r}}$) from the body frame (RSW) to the inertial frame (ECI), and S matrix generates the perturbing acceleration in the body frame from the 14 TFCs. This unknown maneuver representation with 14 TFCs is a mathematical model whose purpose is to rigorously represent the perturbing motion of a satellite. Although it does not recover the actual perturbing acceleration for a given maneuver, it provides a unique dynamics model that can generate an equivalent orbital change for a given unknown maneuver. Therefore, this event representation can be used to take into account the presence of an unknown acceleration in the tracking of a satellite across an unknown maneuver.

2.2. Modified Extended Kalman Filter with TFCs

Tracking a maneuvering satellite using 14 TFCs consists of a modified EKF with a TFC covariance inflation. In the modified EKF, the state vector is expanded to include the 14 TFCs in order to estimate the perturbing thrust acceleration as well as the orbital state. Representing the unknown dynamics with the 14 TFCs, the governing equations of motion can be rewritten with the augmented state:

$$\frac{d}{dt}\begin{bmatrix}\vec{X} \\ \vec{C}\end{bmatrix} = \begin{bmatrix}\vec{F}(\vec{X}) \\ 0_{14\times1}\end{bmatrix} + \begin{bmatrix}B \\ 0_{14\times3}\end{bmatrix} \cdot \vec{a}_u(t) \tag{10}$$

where $\vec{F}$ and B are from Eq. (2). When updating the reference trajectory at the initial time, this modified EKF initializes the 14 TFCs to zeros with nominal TFC covariance matrix, P_{cc0}:

$$Pcc_0 = \lambda\, I_{14\times14} \tag{11}$$

where λ is the norm value of nominal 14 TFCs that is obtained by running this filter over the pre-maneuver tracking arc when getting an initial reference trajectory. The computed norm value, λ, can also be used as a threshold value to determine a maneuver onset and a termination time, which is already shown in Ref. [13]. The modified EKF operates in its normal mode in the absence of any maneuvers and the 14 TFCs works as a process noise to account for any systematic error such as a measurement error. The estimated values of 14 TFCs are so small that these nominal TFCs do not alter the reference trajectory significantly.

When there is an unknown maneuver, the filter recognizes it by detecting any measurement residual exceeding a threshold number of standard deviations from the mean residual for that object. Upon detecting this maneuver onset, the modified EKF adjusts 14 TFCs to avoid filter divergence and to compensate for unknown dynamics by increasing the covariance of the 14 TFCs. Instead of using nominal covariance of 14 TFCs, the filter applies the inflated covariance:

$$Pcc_k = \gamma Pcc_k \tag{12}$$

where Pcc_k is the covariance matrix of the 14 TFCs at the current observation time ($t = t_k$) and γ is a scale up factor, 2. If Pcc_k is too small, the true estimation error of the perturbed orbit state exceeds the uncertainty bound. If Pcc_k is chosen too large, the covariance bound is raised to an unnecessarily large value, which permits larger errors in the estimate of solution. Therefore, the filter keeps increasing Pcc_k until measurement residuals fall within desired uncertainty boundaries, and then this iteration process terminates immediately after that.

Using an adequate Pcc_k from Eq. (12), the updated TFC values provide a sufficient thrust acceleration to bring the estimated orbit trajectory close to the true perturbed orbit. In other words, the estimated TFCs successfully account for the unknown acceleration. Then, the process proceeds to the next observation time ($t = t_{k+1}$) and the reference orbit is updated. The filter reinitializes the 14 TFCs to zeros to process next measurement data. At this stage, the problem is that the filter diverges if the covariance of 14 TFCs is reinitialized to the nominal covariance matrix, P_{cc0}. This divergence occurs because using P_{cc0} does not inflate the state covariance sufficiently to account for the uncertainty contributed by TFCs estimation after the covariance inflation. To avoid this problem, the modified EKF uses the mean covariance matrix for Pcc at reinitialization:

$$Pcc_{k+1} = \frac{Pcc_k + Pcc_0}{2} \tag{13}$$

Even when there is no actual maneuver, this procedure is still valid since Pcc_k would be similar to Pcc_0 and therefore $Pcc_{k+1} \simeq (Pcc_0 + Pcc_0)/2 = Pcc_0$. In this way, the proposed filter achieves a good tracking performance during the non-maneuvering period as well as the maneuvering period, without detecting a maneuver termination time explicitly. This process does not require for the filter to have any a priori knowledge of a maneuver. Therefore, the modified EKF can be applied to a wide variety of maneuvering satellite tracking problems, ranging from an impulsive burn to a continuous low-thrust maneuver.

3. SIMULATIONS AND RESULTS

In order to validate the proposed tracking algorithm with the event representation using 14 TFCs, three simulated maneuver cases are studied. All the simulations are performed on a low-earth orbiting (LEO) satellite, and its initial state is shown in Table 1. The different unknown maneuver cases are shown in Table 2. The ΔV is $10m/s$ for both impulsive and 30 minutes continuous burns. Case 3 is simulated to check the performance of the filter when tracking a continuous low-thrust maneuvering satellite where the maneuver is not terminated. A general 2-body dynamics model with respect to Earth-Centered Inertial (ECI) position-velocity coordinate is considered as the known dynamics for all the cases. The sensor measurement error is imitated by adding white Gaussian noises to the true sensor value, and Matlab is used to implement the algorithm. The modified EKF performs OD with measurement data (range, range-rate) over a duration of 6 hours from a single tracking station at one sample per minute. Assume there is a

Table 1: Initial state of LEO satellite

$h_0(km)$	e	$i(deg)$	$\Omega(deg)$	$\omega(deg)$	$\nu(deg)$
1885	0.1	20	20	20	180

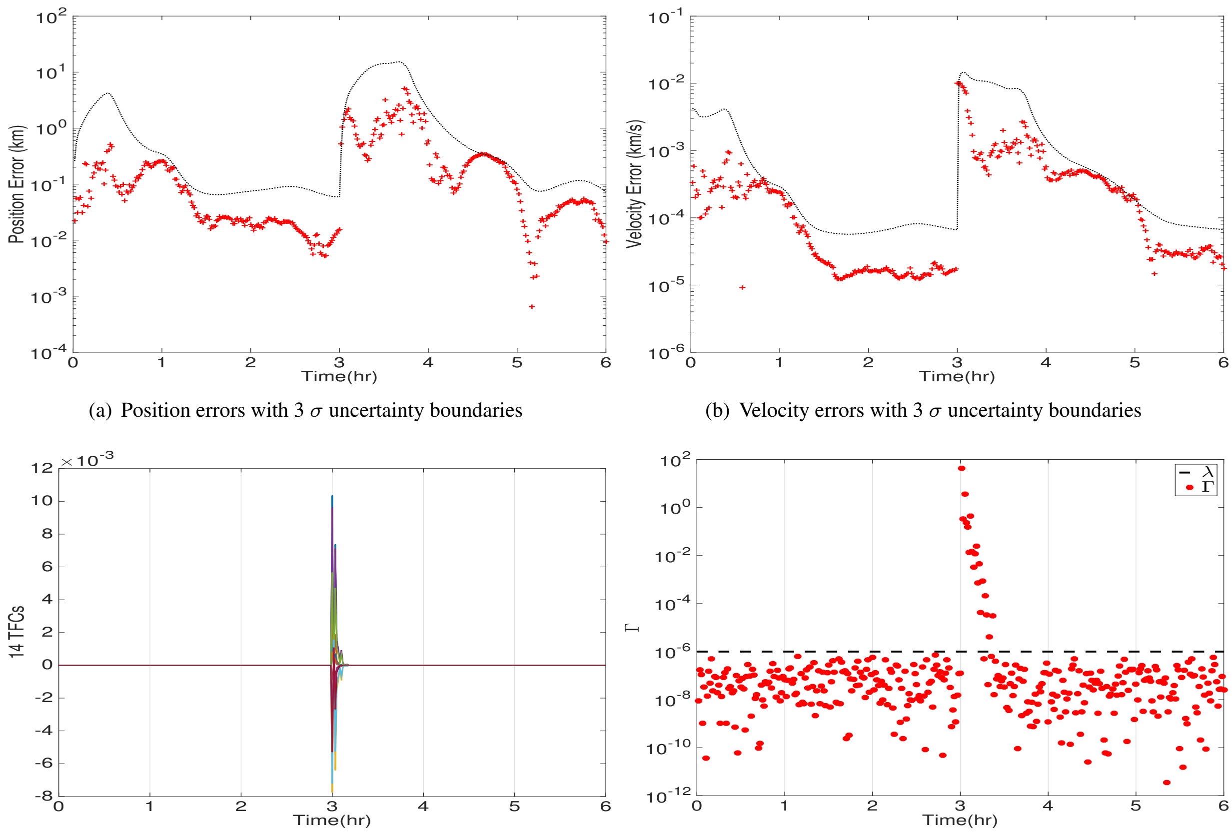
h_0 : altitude at apogee; Earth radius : $6378.137km$[18]

Table 2: Simulation cases

Case	Type	Maneuvering Time
1	maneuver with impulsive burn	$t = 3\ hr$
2	maneuver with 30 minutes continuous burn	$3\ hr < t < 3.5\ hr$
3	low-thrust maneuver with continuous thrusting	$t > 1hr$

fairly accurate a priori orbit information available with pre-maneuver tracking data. The filter utilizes the a priori orbit solution in conjunction with incoming measurement data to maintain tracking of a maneuvering satellite.

Figure 1 shows OD solutions and estimated TFC values from the modified EKF for case 1. The state errors



(a) Position errors with 3 σ uncertainty boundaries

(b) Velocity errors with 3 σ uncertainty boundaries

(c) Time history of TFCs estimation

(d) Generalized norm of estimated 14TFCs

Figure 1: OD solution from the modified EKF for case 1

and uncertainty boundaries decrease until an impulsive burn occurs at $t = 3$hr. Then, the filter increases TFCs to account for the perturbing acceleration, which inflates the uncertainty boundaries to accept the measurement data during the maneuver. The TFCs estimation with an inflated TFC covariance forces measurement residuals fall within 3σ boundaries (30m for range, 3m/s for range-rate), which is shown in Fig. 2. As a result, the state estimation errors are within their estimated 3σ state uncertainty boundaries and the filter is able to maintain tracking of a satellite under an impulsive burn. From Fig. 1(c)-1(d), a maneuver onset time can be easily detected by checking the TFCs estimation or a generalized norm value of 14 TFCs, Γ. There is a change-point where those values increase significantly and go beyond their thresholds. The maneuver is estimated to begin at three hours after the beginning of the simulation. It is also noticeable in Fig. 1(d) that the inflated TFCs falls back to the nominal threshold (λ) at $t = 3.2$hr, which indicates that the maneuver has terminated. This allows us to estimate the duration of an unknown maneuver. Note that the estimation of maneuver termination time from the filter is not accurate mainly due to an inherent difficulty and a delay of detection. However, timely detection of a maneuver termination time is usually not as important as timely detection of maneuver onset time during the tracking. This is because tracking a non-maneuvering satellite assuming it is maneuvering still provides an acceptable state estimation with a related uncertainty boundary [2].

Figure 3 shows that the modified filter can successfully process OD across a maneuver with 30 minutes continuous burn for case 2. The position errors are substantially less than ten kilometers where unknown accelerations do occur, and all state errors are within their estimated 3σ uncertainty boundaries. In Fig. 3(c), there are two big jumps in TFCs estimation, which is different from the impulsive burn case that has a single jump. From this figure, it is difficult to tell whether these two jumps are caused by two consecutive maneuvers or a single maneuver with a longer duration. In this case, checking a generalized norm value is more useful to estimate the maneuver onset and the termination

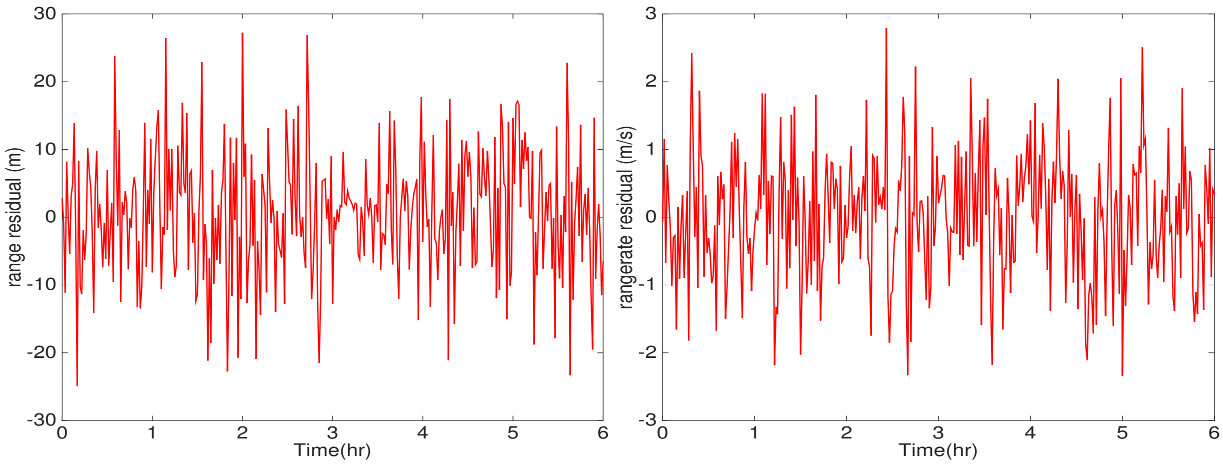
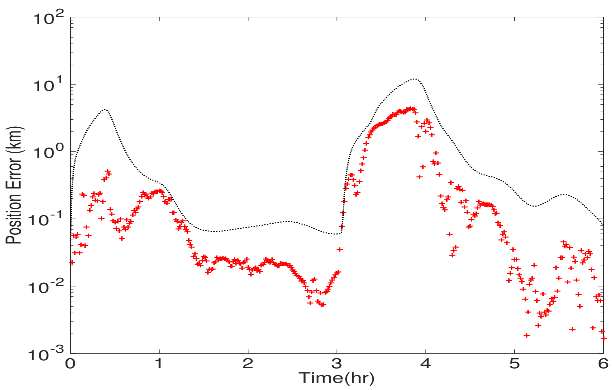
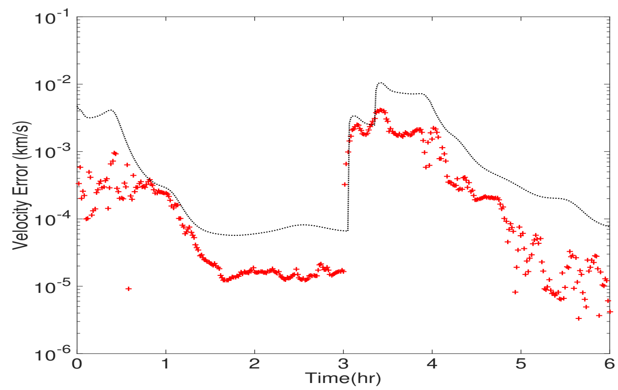


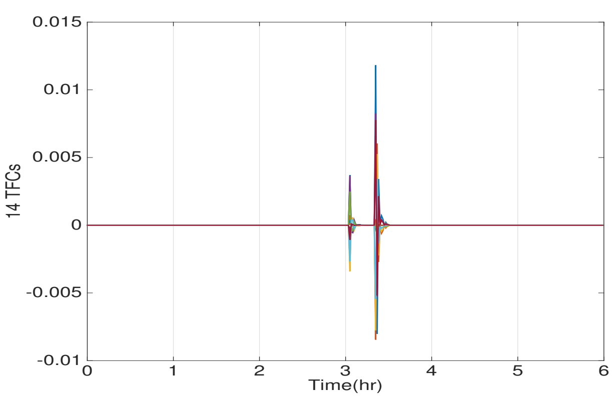
Figure 2: Measurement residuals from the modified EKF for case 1



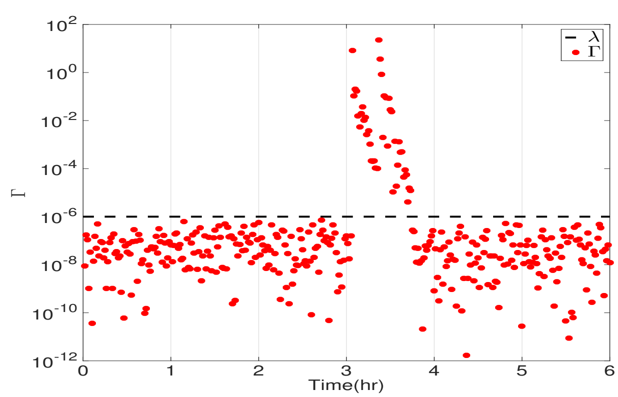
(a) Position errors with 3 σ uncertainty boundaries



(b) Velocity errors with 3 σ uncertainty boundaries



(c) Time history of TFCs estimation

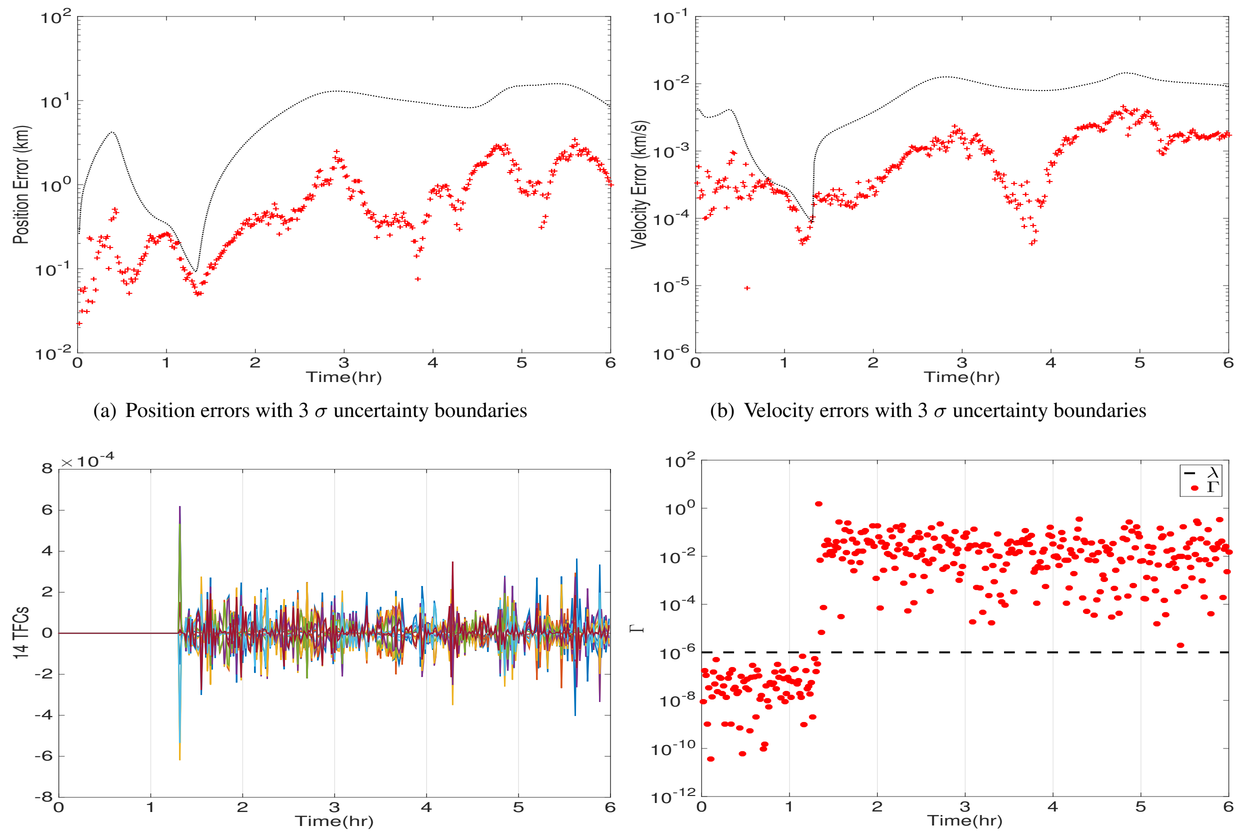


(d) Generalized norm of estimated 14TFCs

Figure 3: OD solution from the modified EKF for case 2

time since it is easier to distinguish those change-points. The actual burn time is thirty minutes while the estimated maneuvering time is about 45 minutes due to the detection delay of termination time. However, the filter is able to maintain tracking of a satellite while providing a valid OD solution during the maneuver and after the maneuver.

Figure 4 shows that the filter detects continuous perturbing accelerations consistently. Once the estimated maneuver begins, the state covariance matrix is inflated so that the filter is able to accept the subsequent measurement data and to provide valid OD solutions. Different from the previous cases, there is no maneuver termination time for



(a) Position errors with 3 σ uncertainty boundaries

(b) Velocity errors with 3 σ uncertainty boundaries

(c) Time history of TFCs estimation

(d) Generalized norm of estimated 14TFCs

Figure 4: OD solution from the modified EKF for case 3

case 3, which explains why those inflated TFCs do not decrease back to the nominal values in Fig. 4(c) and 4(d). The low-thrust burn happens at one hour after the initial time, but the algorithm estimates that the maneuver begins after $t = 1.15$hr. There is a detection delay of around fifteen minutes, which is due to the fact that the accumulated effect of low-thrust burn does not show up immediately in the estimation process. From all the simulated cases, the proposed maneuver tracking algorithm shows consistent performance of providing valid OD solutions across unknown maneuvers and continues tracking of satellites. Aside from successfully tracking a satellite, the proposed algorithm also provides a direct estimate of the unknown maneuvering period.

4. CONCLUSION

Tracking of maneuvering satellites based on an unknown maneuver representation with thrust Fourier coefficient (TFC) is studied. A modified extended Kalman filter (EKF) is developed and its tracking performance is tested by using different maneuver simulation studies. The simulation result shows that the modified EKF is able to maintain tracking of a satellite by providing valid orbit solutions across unknown maneuvers. This EKF has three main advantages. First, unlike other methods, no pre-defined model of unknown accelerations is required. Second, it is a fully automated process that is also simple to implement. Third, it is able to track a wide range of maneuvering satellites as they undergo maneuvers, ranging from an impulsive burn to a continuous low-thrust burn. In addition, the filter also enables us to estimate unknown maneuver onset and termination time directly from observation data.

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