Addressing The Up-Link Problem Using RytovProp

David L. Fried

un affiliated

RytovProp is a new concept for the simulation of optical propagation through atmospheric turbulence. The concept is based on a felicitous combination of Rytov approximation based analytic results, matrix theory, and gaussian random variable theory. Though less versatile than a wave optics propagation simulation approach, for those problems which it can accommodate it provides large sets of results orders of magnitude faster than a wave optics propagation simulation. In this work I describe the RytovProp concept and show results demonstrating its applicability to the Up-Link problem.

1. Introduction

I use the term "Up-Link" to refer to a ground-to-space optical communications system—a system that establishes a link from a ground based laser transmitter to a receiver on an orbiting satellite. The laser transmitter has a relatively small diameter and no higher-order adaptive optics, only some limited tip/tilt correction capability. I take as the "Up-Link Problem" the task of developing statistical results characterizing the effects of atmospheric turbulence on the signal strength, or Strehl ratio, of such a link—in particular the task of developing results for the Strehl ratio's probability distribution. To address this problem I have developed a new computational procedure which, because it relies on results developed using the Rytov approximation to capture the optical propagation effects of atmospheric turbulence, I call RytovProp.

The RytovProp technique is a Monte Carlo approach to the Up-Link problem. It relies on the generation of a very large number of statistically appropriate, randomly selected realizations of the value of the Strehl ratio. From these the statistics of the Up-Link are calculated. The RytovProp computational procedure allows hundreds of thousands of statistically independent realizations of the value of the Strehl ratio to be generated on a PC in just a few seconds. This great speed is made possible by a judicious exploitation of propagation theory analytic results, a bit of matrix theory, and use of gaussian random variable theory.

2. Calculating Signal Strength: Problem Formulation

I consider a laser transmitter system with a circular aperture of diameter D, and define a square-patterned lattice of points on the aperture plane, an array of points sufficiently dense that it can be considered to adequately sample the aperture. I chose the spacing between adjacent lattice points to be such that there are P lattice-points within the aperture, denoting their positions by $\mathbf{r}_p = \left(x_p, y_p\right)$ where $p = \{1,\ 2,\ 3,\ \ldots,\ P\}$. I shall use the notation $\mathbf{r}_0$ to denote the position of the point at the center of the aperture. The placement of this lattice of points is such that it is symmetric about this center-point but does not include that center-point—so the center-point (at $\mathbf{r}_0$) will be in the center of the small square formed by the four points of the lattice that are closest to $\mathbf{r}_0$.

Whether speaking of the laser beam transmitted to the satellite or of the tip/tilt measurement signal from a beacon on the satellite, in referring to the associated time I shall always have in mind the time at the ground—the time when the signal left the transmitter or when the beacon signal arrived at the tip/tilt tracker. Farther more, I shall assume that the time of flight of light through the atmospheric portion of the propagation path is negligibly short, so even when speaking of the time dependence of the turbulence pattern at some position along the propagation path the term "time" always refers to the associated instant of time at the ground site.

I use the notation t_0 to denote what I call the "current time," the time associated with some signal for which the Strehl ratio, $\mathcal{S}$, is to be evaluated. I shall consider a sequence of M prior times, t_m, where $m = \{1,\ 2,\ 3,\ \ldots,\ M\}$ —a sequence of tip/tilt measurement times. Associated with these times is a sequence of nominal propagation directions, $\boldsymbol{\theta}_m$, where $m = \{0,\ 1,\ 2,\ 3,\ \ldots\ ,\ M\}$. I note that $\boldsymbol{\theta}_m$ is directly proportional to t_m —except that $\boldsymbol{\theta}_0$, since it is associated with transmission of the laser beam rather than with receipt of the beacon signal, deviates from this simple linear dependence by the point-ahead angular distance.

Associated with the $(M{+}1)\times(P{+}1)$ combinations of time, t_m, (and associated propagation direction, $\boldsymbol{\theta}_m$) with position, $\mathbf{r}_p$, I use the notation $\phi_{p,m}$ and $\ell_{p,m}$ to denote the turbulence induced perturbation of the phase and of the log of the amplitude respectively of the optical field in propagation between the satellite and the aperture plane of the laser transmitter. It is to be noted that, by virtue of the reciprocity principle in optical propagation between two points, it is not necessary to indicate which is the source point; the same values for the phase and log-amplitude perturbations apply which ever one of the two points is the source point. (The quantities $\phi_{p,m}$ and $\ell_{p,m}$ apply equally well to perturbation of light going up and of light coming down.)

The essence of the RytovProp method lies in the following statement:

> If a statistically appropriate, randomly selected realization of the set of $P \times (M+1)$ values for the turbulence induced phase perturbations, $\phi_{p,m}$, and for the set of P values for the turbulence induced log-amplitude perturbations, $\ell_{p,0}$, were available then the value of the Strehl ratio, $\mathcal{S}$, associated with the signal strength could be calculated according to the formula that
>
> $$\mathcal{S} = \left| \sum_{p=1}^{P} A_p\, \Delta^2 \exp\Big(\ell_{p,0} + i\,[\,\phi_{p,0} - \varphi_p\,] \Big) \right|^2 \Big/ \left| \sum_{p=1}^{P} A_p\, \Delta^2 \right|^2 , \tag{1}$$
>
> where A_p represents the amplitude of the laser beam at position $\mathbf{r}_p$ on the transmitter's aperture, Δ represents the distance between adjacent points in the set of P points covering the transmitter's aperture, and φ_p represents the optical system imposed phase shift at position $\mathbf{r}_p$ on the transmitter's aperture plane that is to be associated with tip/tilt (or adaptive optics) corrections.

The RytovProp method functions by (in essence) being able to develop a statistically appropriate, randomly selected set of values for all of the $\phi_{p,m}$, $\ell_{p,0}$, and φ_p quantities, and then using this equation to obtain a statistically appropriate, randomly selected value for the Strehl ratio, $\mathcal{S}$.

For the type of laser transmitter system being considered here, for which there is only tip/tilt correction (and no higher-order adaptive optics correction) the value of φ_p is given by the equation

$$\varphi_p = k\,(\widetilde{x}_p\, \vartheta^{\mathrm{X}} + \widetilde{y}_p\, \vartheta^{\mathrm{Y}})\,, \tag{2}$$

where $k = 2\,\pi/\lambda$, the quantities ϑ^{X} and ϑ^{Y} denote the two components of tip/tilt correction applied by the laser transmitter system at the current time, t_0, and

$$\widetilde{x}_p = \frac{x_p - \overline{x}}{\sqrt{\displaystyle\sum_{p'=1}^{P} \big(x_{p'} - \overline{x}\big)^2}}\,, \qquad \text{and} \qquad \widetilde{y}_p = \frac{y_p - \overline{y}}{\sqrt{\displaystyle\sum_{p'=1}^{P} \big(y_{p'} - \overline{y}\big)^2}}\,, \tag{3}$$

with

$$\overline{x} = \sum_{p=1}^{P} x_p\,, \qquad \text{and} \qquad \overline{y} = \sum_{p=1}^{P} y_p\,. \tag{4}$$

It is to be noticed that

$$\sum_{p=1}^{P} \widetilde{x}_p = 0\,, \qquad \sum_{p=1}^{P} \widetilde{y}_p = 0\,, \qquad \sum_{p=1}^{P} \widetilde{x}_p^{\,2} = 1\,, \qquad \text{and} \qquad \sum_{p=1}^{P} \widetilde{y}_p^{\,2} = 1\,. \tag{5}$$

With the quantities ϑ^{X}_{m} and ϑ^{Y}_{m} denoting the two components of tip/tilt measured at prior time t_{m}, the values of the two components of tip/tilt correction, ϑ^{X} and ϑ^{Y}, are given by the equation

$$\vartheta^{X} = \sum_{m=1}^{M} \alpha_{m}\, \vartheta^{X}_{m} \,, \qquad \text{and} \qquad \vartheta^{Y} = \sum_{m=1}^{M} \alpha_{m}\, \vartheta^{Y}_{m} \,, \tag{6}$$

where the set of M different values of α_{m} represent a set of coefficients that may be thought of as defining the bandwidth of the beacon-tip/tilt tracking servo system. The prior time components of tip/tilt, ϑ^{X}_{m} and ϑ^{Y}_{m}, may be considered to have values given by the equation

$$\vartheta^{X}_{m} = k^{-1} \sum_{p=1}^{P} \widetilde{x}_{p}\, \phi_{p,m} \,, \qquad \text{and} \qquad \vartheta^{Y}_{m} = k^{-1} \sum_{p=1}^{P} \widetilde{y}_{p}\, \phi_{p,m} \,. \tag{7}$$

At this point it is convenient to introduce the notations $\widetilde{\phi}_{p,m}$ and $\widetilde{\ell}_{p,0}$ which are defined by the equation

$$\widetilde{\phi}_{p,m} = \phi_{p,m} - \phi_{0,m} \,, \qquad \text{and} \qquad \widetilde{\ell}_{p,0} = \ell_{p,0} - \overline{\ell} \,, \tag{8}$$

with $\overline{\ell}$ denoting the mean (*i.e.* the ensemble average) value of the turbulence induced log-amplitude perturbations, which mean value is independent of the aperture plane position, $\mathbf{r}_{p}$. In distinction to the phase and log-amplitude perturbations, $\phi_{p,m}$ and $\ell_{p,m}$, I refer to these quantities, *i.e.* to $\widetilde{\phi}_{p,m}$ and $\widetilde{\ell}_{p,m}$, as the adjusted phase and log-amplitude perturbations. Making use of this notation I can recast Eq.'s (1) and (7) as

$$\mathcal{S} == \left| \sum_{p=1}^{P} A_{p}\, \exp\Big([\, \widetilde{\ell}_{p,0} + \overline{\ell}\,] + i\, [\, \widetilde{\phi}_{p,0} - \varphi_{p}\,] \Big) \right|^{2} \Bigg/ \left| \sum_{p=1}^{P} A_{p} \right|^{2} , \tag{9}$$

and

$$\vartheta^{X}_{m} = k^{-1} \sum_{p=1}^{P} \widetilde{x}_{p}\, \widetilde{\phi}_{p,m} \,, \qquad \text{and} \qquad \vartheta^{Y}_{m} = k^{-1} \sum_{p=1}^{P} \widetilde{y}_{p}\, \widetilde{\phi}_{p,m} \,. \tag{10}$$

I now rephrase the statement concerning the essence of the RytovProp method, saying:

> If a statistically appropriate, randomly selected realization of the set of $P \times (M+1)$ values for $\widetilde{\phi}_{p,m}$ and for the set of P values for $\widetilde{\ell}_{p,0}$ were available, then the value of the Strehl ratio, $\mathcal{S}$, could be calculated using Eq.'s (9) and (10).

In the next section I take up the matter of establishing a basis for quickly generating a statistically appropriate, randomly selected realization of this set of values.

3. Generating A Statistically Appropriate, Random Realization

I separate this section into three quite distinct sub sections. In the first of these I develop a matrix theory based result concerning the generation of a set of jointly gaussian random variables. In the second of these I present propagation theory related results for the phase and log-amplitude perturbations, $\phi_{p,m}$ and $\ell_{p,m}$, results derived using the Rytov approximation. In the third sub section I show how these results can be adapted so they apply for the adjusted phase and log-amplitude perturbations, $\widetilde{\phi}_{p,m}$ and $\widetilde{\ell}_{p,m}$. Together theses three sub sections provide a basis for quickly generating a statistically appropriate, randomly selected realization of the needed set of values for $\widetilde{\phi}_{p,m}$ and for $\widetilde{\ell}_{p,0}$.

3.1 Producing A Set Of Jointly Gaussian Random Variables

Let z_{n} denote a set of $n = \{1,\, 2,\, 3,\, \ldots\,,\; N\}$ zero-mean, jointly-gaussian, random variables, and let $\mathbf{z}$ denote a column vector whose n^{TH} element is z_{n}. Using the angle-bracket notation, $\langle \ldots \rangle$, to indicate an ensemble average the covariance matrix, $\mathbf{C}_z$, for this set of random variables can be seen to be given by the equation

$$\mathbf{C}_z = \langle\, \mathbf{z}\,\mathbf{z}^{\mathrm{T}} \,\rangle\,. \tag{11}$$

This is an N-by-N size matrix.

Let $\mathbf{U}_z$ denote a matrix whose columns correspond to the eigen-vectors of $\mathbf{C}_z$, and let $\mathbf{S}_z$ denote a diagonal matrix whose diagonal elements are the corresponding eigen-values of $\mathbf{C}_z$. Both $\mathbf{U}_z$ and $\mathbf{S}_z$ are, like $\mathbf{C}_z$, N-by-N size matrices. I can write

$$\mathbf{C}_z\,\mathbf{U}_z = \mathbf{U}_z\,\mathbf{S}_z\,. \tag{12}$$

Because the eigen vectors are ortho-normal $\mathbf{U}_z$ is unitary/orthogonal matrix and I can write

$$\mathbf{U}_z^{\mathrm{T}}\,\mathbf{U}_z = \mathbf{I}\,, \qquad \text{as well as} \qquad \mathbf{U}_z\,\mathbf{U}_z^{\mathrm{T}} = \mathbf{I}\,, \tag{13}$$

where $\mathbf{I}$ denotes the identity matrix of size N-by-N —a diagonal matrix all of whose diagonal elements are equal to unity.

Let $\widetilde{\mathbf{S}}_z$ denote a diagonal matrix whose diagonal elements are each equal to the square root of the corresponding one of the diagonal elements of $\mathbf{S}_z$, so that $\widetilde{\mathbf{S}}_z\,\widetilde{\mathbf{S}}_z^{\mathrm{T}} = \mathbf{S}_z$. Now form the matrix $\widetilde{\mathbf{C}}_z$ according to the equation

$$\widetilde{\mathbf{C}}_z = \mathbf{U}_z\,\widetilde{\mathbf{S}}_z\,. \tag{14}$$

It is easy to shown that this matrix, $\widetilde{\mathbf{C}}_z$, is the square root of the matrix $\mathbf{C}_z$ —the square root of $\mathbf{C}_z$ in the sense that $\widetilde{\mathbf{C}}_z\,\widetilde{\mathbf{C}}_z^{\mathrm{T}} = \mathbf{C}_z$. To prove this—successively making use of the fact that $\widetilde{\mathbf{S}}_z\,\widetilde{\mathbf{S}}_z^{\mathrm{T}} = \mathbf{S}_z$ and of Eq. (14), and then (13) —I can write

$$\widetilde{\mathbf{C}}_z\,\widetilde{\mathbf{C}}_z^{\mathrm{T}} = \big(\mathbf{U}_z\,\widetilde{\mathbf{S}}_z\big)\,\big(\mathbf{U}_z\,\widetilde{\mathbf{S}}_z\big)^{\mathrm{T}} = \mathbf{U}_z\,\widetilde{\mathbf{S}}_z\,\widetilde{\mathbf{S}}_z^{\mathrm{T}}\,\mathbf{U}_z^{\mathrm{T}} = \mathbf{U}_z\,\mathbf{S}_z\,\mathbf{U}_z^{\mathrm{T}} = \mathbf{C}_z\,\mathbf{U}_z\,\mathbf{U}_z^{\mathrm{T}} = \mathbf{C}_z\,. \tag{15}$$

To generate a single realization I start by letting $\boldsymbol{\gamma}$ be a column vector of length N of randomly selected zero-mean, unity-variance, statistically-independent, gaussian random variables—so that

$$\langle\, \boldsymbol{\gamma} \,\rangle = \mathbf{0}\,, \qquad \text{and} \qquad \langle\, \boldsymbol{\gamma}\,\boldsymbol{\gamma}^{\mathrm{T}} \,\rangle = \mathbf{I}\,, \tag{16}$$

where $\mathbf{0}$ is a column vector of length N all of whose elements are equal to zero and, as above, $\mathbf{I}$ is the identity matrix of size N-by-N.

I assert that the column vector $\boldsymbol{\zeta}$ formed according to the equation

$$\boldsymbol{\zeta} = \widetilde{\mathbf{C}}_z\,\boldsymbol{\gamma}\,, \tag{17}$$

is a statistically appropriate realization of the random variable $\mathbf{z}$. To prove that this is so I first note that since the elements of $\boldsymbol{\gamma}$ taken as a set of random variables constitute a realizations of a jointly gaussian set of random variables, and since the elements of $\boldsymbol{\zeta}$ are formed as a weighted sums of the elements of $\boldsymbol{\gamma}$, then the elements of $\boldsymbol{\zeta}$ constitute a realization of a set of jointly gaussian random variables. I next note that since $\langle\, \boldsymbol{\gamma} \,\rangle = 0$, then since $\widetilde{\mathbf{C}}_z$ is non random then

$$\langle\, \boldsymbol{\zeta} \,\rangle = \langle\, \widetilde{\mathbf{C}}_z\,\boldsymbol{\gamma} \,\rangle = \widetilde{\mathbf{C}}_z\,\langle\, \boldsymbol{\gamma} \,\rangle = \widetilde{\mathbf{C}}_z\,\mathbf{0} = \mathbf{0}\,. \tag{18}$$

Finally, I note that

$$\langle\, \boldsymbol{\zeta}\,\boldsymbol{\zeta}^{\mathrm{T}} \,\rangle = \Big\langle\, \big(\widetilde{\mathbf{C}}_z\,\boldsymbol{\gamma}\big)\,\big(\widetilde{\mathbf{C}}_z\,\boldsymbol{\gamma}\big)^{\mathrm{T}} \,\Big\rangle = \Big\langle\, \widetilde{\mathbf{C}}_z\,\boldsymbol{\gamma}\,\boldsymbol{\gamma}^{\mathrm{T}}\,\widetilde{\mathbf{C}}_z^{\mathrm{T}} \,\Big\rangle = \widetilde{\mathbf{C}}_z\,\langle\, \boldsymbol{\gamma}\,\boldsymbol{\gamma}^{\mathrm{T}} \,\rangle\,\widetilde{\mathbf{C}}_z^{\mathrm{T}} = \widetilde{\mathbf{C}}_z\,\mathbf{I}\,\widetilde{\mathbf{C}}_z^{\mathrm{T}} = \widetilde{\mathbf{C}}_z\,\widetilde{\mathbf{C}}_z^{\mathrm{T}} = \mathbf{C}_z\,. \tag{19}$$

These three just noted fact indicate that the values of the elements of $\boldsymbol{\zeta}$, calculated in accordance with Eq. (17), are a statistically appropriate realization of the random variable elements of the column vector $\mathbf{z}$. They have the correct mean value, the correct set of covariances and cross-covariances, and are jointly gaussian.

From consideration of what has just been shown it is clear that if we had, in numerical form, the covariance matrix relating all of the phase and log-amplitude perturbations we could develop the square root of that matrix and then, by simply multiplying that square root matrix by a column vector of statistically independent normal random variables, could generate a randomly selected realization of the value of those phases and log-amplitudes—and from those phase and log-amplitude values could develop a randomly selected statistically appropriate value for the Strehl ratio. As a first step in developing that matrix, in the next sub section I briefly review/present, in analytic form, the relevant propagation results, developed using the Rytov approximation. These analytic results will allow me to carry out the calculation of the values of the elements of that covariance matrix.

3.2 Rytov Approximation Based Analytic Propagation Results

Based on use of the Rytov approximation results have been developed allowing the numerical evaluation of the log-amplitude covariance, $\mathcal{C}_{\ell\ell}(p,m;p',m)$, for the phase structure function, $\mathcal{D}_{\phi\phi}(p,m;p',m')$, and for the phase:log-amplitude cross-covariance, $\mathcal{C}_{\phi\ell}(p,m;p',m')$ —which quantities are defined by the equation

$$\mathcal{C}_{\ell\ell}(p,m;p',m') = \Big\langle \big[\, \ell_{p,m} - \overline{\ell}\,\big] \big[\, \ell_{p',m} - \overline{\ell}\,\big] \Big\rangle , \tag{20a}$$

$$\mathcal{D}_{\phi\phi}(p,m;p',m') = \Big\langle \big[\, \phi_{p,m} - \phi_{p',m'}\,\big]^2 \Big\rangle , \tag{20b}$$

$$\mathcal{C}_{\phi\ell}(p,m;p',m') = \Big\langle \phi_{p,m} \big[\, \ell_{p',m'} - \overline{\ell}\,\big] \Big\rangle . \tag{20c}$$

Making use of the Rytov approximation it has been shown that the values of these quantities are given by the equations

$$\mathcal{C}_{\ell\ell}(p,m;p',m') = \frac{8.16}{4\,\pi}\,k^2 \int_0^Z dz\, C_{\mathrm{N}}^2(z)\,\big[\, z\,(1-z/Z)/k\,\big]^{5/6} F\big(Q(p,m;p',m';z)\big) , \tag{21a}$$

$$\mathcal{D}_{\phi\phi}(p,m;p',m') = \frac{8.16}{2\,\pi}\,k^2 \int_0^Z dz\, C_{\mathrm{N}}^2(z)\,\big[\, z\,(1-z/Z)/k\,\big]^{5/6} G\big(Q(p,m;p',m';z)\big) , \tag{21b}$$

$$\mathcal{C}_{\phi\ell}(p,m;p',m') = \frac{8.16}{4\,\pi}\,k^2 \int_0^Z dz\, C_{\mathrm{N}}^2(z)\,\big[\, z\,(1-z/Z)/k\,\big]^{5/6} H\big(Q(p,m;p',m';z)\big) , \tag{21c}$$

where Z denotes the length of the propagation path and z, the variable of integration, denotes position along the propagation path (with $z = 0$ corresponding to the ground end of the path), where the three functions $F(Q)$, $G(Q)$, and $H(Q)$ are defined by the equations

$$F(Q) = \int_0^\infty d\kappa\, \kappa^{-8/3}\, \mathrm{J}_0\big(\kappa\, Q\big) \Big[\, 1 - \cos(\kappa^2) \Big] , \tag{22a}$$

$$G(Q) = \int_0^\infty d\kappa\, \kappa^{-8/3} \Big[\, 1 - \mathrm{J}_0\big(\kappa\, Q\big) \Big] \Big[\, 1 + \cos(\kappa^2) \Big] , \tag{22b}$$

$$H(Q) = \int_0^\infty d\kappa\, \kappa^{-8/3}\, \mathrm{J}_0\big(\kappa\, Q\big)\, \sin(\kappa^2) . \tag{22c}$$

and where the quantity $Q(p,m;p',m';z)$ that appears in Eq. (21) has a value given by the equation

$$Q(p,m;p',m';z) = \big|\, (\mathbf{r}_p - \mathbf{r}_{p'})\,(1-z/Z) + (\boldsymbol{\theta}_m - \boldsymbol{\theta}_{m'})\,z - \mathbf{V}(z)\,(t_m - t_{m'})\,\big|\, \mathcal{L}_{\mathrm{F}}^{\;-1} . \tag{23}$$

Here the quantity $\mathbf{V}(z)$ denotes the projection onto a plane parallel to the aperture plane of the vector representing the wind velocity that exists at the range z, and the quantity $\mathcal{L}_{\mathrm{F}}$, which may be considered to be a sort of Fresnel length, has a value given by the equation

$$\mathcal{L}_{\mathrm{F}} = \big[\, z\,(Z-z) \big/ (k\,Z)\,\big]^{-1/2} . \tag{24}$$

Evaluation of the integrals in Eq. (22) yield the results that

$$F(Q) = \tfrac{1}{2}\,\frac{\Gamma\big(-\tfrac{5}{6}\big)}{\Gamma\big(\tfrac{11}{6}\big)}\,\big(\tfrac{1}{4}\,Q^2\big)^{5/6} - \tfrac{1}{2}\,\Gamma\big(-\tfrac{5}{6}\big)\,\Re\Big\{\,\exp\big(\tfrac{5}{12}\,\pi\,i\big)\,{}_1F_1\big(-\tfrac{5}{6};1;\tfrac{1}{4}\,Q^2\,i\big)\,\Big\}\,, \tag{25a}$$

$$G(Q) = -\tfrac{1}{2}\,\frac{\Gamma\big(-\tfrac{5}{6}\big)}{\Gamma\big(\tfrac{11}{6}\big)}\,\big(\tfrac{1}{4}\,Q^2\big)^{5/6} - \tfrac{1}{2}\,\Gamma\big(-\tfrac{5}{6}\big)\,\Big[\,\Re\Big\{\,\exp\big(\tfrac{5}{12}\,\pi\,i\big)\,{}_1F_1\big(-\tfrac{5}{6};1;\tfrac{1}{4}\,Q^2\,i\big)\,\Big\} - \cos\big(\tfrac{5}{12}\,\pi\big)\,\Big]\,, \tag{25b}$$

$$H(Q) = \tfrac{1}{2}\,\Gamma\big(-\tfrac{5}{6}\big)\,\Re\Big\{\,\exp\big(\tfrac{11}{12}\,\pi\,i\big)\,{}_1F_1\big(-\tfrac{5}{6};1;\tfrac{1}{4}\,Q^2\,i\big)\,\Big\}\,. \tag{25c}$$

The hyper geometric functions appearing in this result can be evaluated using the standard power series formulation—so long as Q is not too large. With 16-digit computational accuracy quite accurate results can be developed for values of Q as large as $Q = 10$, but for values of Q much larger than about $Q = 12$ the results obtained with 16-digit computational accuracy are very clearly in error. For values of Q larger than $Q = 10$ I have developed the asymptotic series results that

$$\begin{aligned} F(Q) = {} & \tfrac{1}{4}\,\frac{\Gamma\big(\tfrac{7}{6}\big)}{\Gamma\big(-\tfrac{1}{6}\big)}\,\big(\tfrac{1}{4}\,Q^2\big)^{-7/6} - \tfrac{1}{2}\,\big(\tfrac{1}{4}\,Q^2\big)^{-11/6}\Big[\,1 - \tfrac{8}{3}\,\tfrac{11}{3}\,\big(\tfrac{1}{4}\,Q^2\big)^{-2} + \tfrac{8}{3}\,\tfrac{11}{3}\,\tfrac{14}{3}\,\tfrac{17}{3}\,\big(\tfrac{1}{4}\,Q^2\big)^{-4} - \,\ldots\,\Big]\,\sin\big(\tfrac{1}{4}\,Q^2\big) \\ & + \tfrac{1}{2}\,\big(\tfrac{1}{4}\,Q^2\big)^{-17/6}\Big[\,\tfrac{8}{3} - \tfrac{8}{3}\,\tfrac{11}{3}\,\tfrac{14}{3}\,\big(\tfrac{1}{4}\,Q^2\big)^{-2} + \tfrac{8}{3}\,\tfrac{11}{3}\,\tfrac{14}{3}\,\tfrac{17}{3}\,\tfrac{20}{3}\,\big(\tfrac{1}{4}\,Q^2\big)^{-4} - \,\ldots\,\Big]\,\cos\big(\tfrac{1}{4}\,Q^2\big)\,, \end{aligned} \tag{26a}$$

$$G(Q) = -\frac{\Gamma\big(-\tfrac{5}{6}\big)}{\Gamma\big(\tfrac{11}{6}\big)}\,\big(\tfrac{1}{4}\,Q^2\big)^{5/6} + F(Q) + \tfrac{1}{2}\,\Gamma\big(-\tfrac{5}{6}\big)\,\cos\big(\tfrac{5}{12}\,\pi\big)\,, \tag{26b}$$

$$\begin{aligned} H(Q) = {} & \tfrac{1}{2}\,\frac{\Gamma\big(\tfrac{1}{6}\big)}{\Gamma\big(\tfrac{5}{6}\big)}\,\big(\tfrac{1}{4}\,Q^2\big)^{-1/6} + \tfrac{1}{2}\,\big(\tfrac{1}{4}\,Q^2\big)^{-11/6}\Big[\,1 - \tfrac{8}{3}\,\tfrac{11}{3}\,\big(\tfrac{1}{4}\,Q^2\big)^{-2} + \tfrac{8}{3}\,\tfrac{11}{3}\,\tfrac{14}{3}\,\tfrac{17}{3}\,\big(\tfrac{1}{4}\,Q^2\big)^{-4} - \,\ldots\,\Big]\,\cos\big(\tfrac{1}{4}\,Q^2\big) \\ & + \tfrac{1}{2}\,\big(\tfrac{1}{4}\,Q^2\big)^{-17/6}\Big[\,\tfrac{8}{3} - \tfrac{8}{3}\,\tfrac{11}{3}\,\tfrac{14}{3}\,\big(\tfrac{1}{4}\,Q^2\big)^{-2} + \tfrac{8}{3}\,\tfrac{11}{3}\,\tfrac{14}{3}\,\tfrac{17}{3}\,\tfrac{20}{3}\,\big(\tfrac{1}{4}\,Q^2\big)^{-4} - \,\ldots\,\Big]\,\sin\big(\tfrac{1}{4}\,Q^2\big)\,, \end{aligned} \tag{26c}$$

which results very smoothly joint the results given by Eq. (25) at $Q = 10$.

I add to these basic propagation theory results the result that

$$\overline{\ell} = -\,\mathcal{C}_{\ell\ell}(p,m;p,m) = -\mathcal{R}y\,, \tag{27}$$

where the notation $\mathcal{R}y$ is used to denote what has come to be called the Rytov-number.

These results are all for the statistics of the phase and log-amplitude perturbations, $\phi_{p,m}$ and $\ell_{p,m}$. What is needed is results for the adjusted phase and log-amplitude perturbations, $\widetilde{\phi}_{p,m}$ and $\widetilde{\ell}_{p,m}$. In the next section I show how such results can be obtained from the results presented here.

3.3 Adapting Phase And Log-Amplitude Results To Apply To The Adjusted Phase And Log-Amplitude

The elements of the covariance matrix are the covariance of the adjusted phase, $\mathcal{C}_{\tilde\phi\tilde\phi}(p,m;p',m')$, the covariance of the adjusted log-amplitude, $\mathcal{C}_{\tilde\ell\tilde\ell}(p,m;p',m')$, and the cross-covariance between the adjusted phase and the adjusted log-amplitude, $\mathcal{C}_{\tilde\phi\tilde\ell}(p,m;p',m')$. These quantities are expressible as

$$\mathcal{C}_{\tilde\phi\tilde\phi}(p,m;p',m') = \big\langle\,\widetilde{\phi}_{p,m}\,\widetilde{\phi}_{p',m'}\,\big\rangle\,,\qquad \mathcal{C}_{\tilde\ell\tilde\ell}(p,m;p',m') = \big\langle\,\widetilde{\ell}_{p,m}\,\widetilde{\ell}_{p',m'}\,\big\rangle\,,$$
$$\text{and}\quad \mathcal{C}_{\tilde\phi\tilde\ell}(p,m;p',m') = \big\langle\,\widetilde{\phi}_{p,m}\,\widetilde{\ell}_{p'm'}\,\big\rangle\,. \tag{28}$$

Noting that $(a-b)\,(c-d) = \frac{1}{2}\,\big[\,-(a-c)^2 + (a-d)^2 + (b-c)^2 - (b-d)^2\,\big]$ and making use of Eq.'s (8) and (20b) it can be seen that

$$\mathcal{C}_{\tilde\phi\tilde\phi}(p,m;p',m') = \tfrac{1}{2}\,\big[\,-\mathcal{D}_{\phi\phi}(p,m;p',m') + \mathcal{D}_{\phi\phi}(p,m;0,m') + \mathcal{D}_{\phi\phi}(0,m;p',m') - \mathcal{D}_{\phi\phi}(0,m;0,m')\,\big]\,. \tag{29}$$

From Eq.'s (8) and (20a) it follows that

$$\mathcal{C}_{\tilde\ell\tilde\ell}(p,m;p',m') = \mathcal{C}_{\ell\ell}(p,m;p',m')\,. \tag{30}$$

And finally, from Eq.'s (8) and (20c) it can be seen that

$$\mathcal{C}_{\tilde{\phi}\tilde{\ell}}(p,m;p',m') = \mathcal{C}_{\phi\ell}(p,m;p',m') - \mathcal{C}_{\phi\ell}(0,m;p',m')\,. \tag{31}$$

With all of these equations we are able to populate the covariance matrix with numerical values, and could directly proceed to the numerical computation of Strehl ratio values. However, as is developed in the next section, we can greatly reduce the size of the computational task if we directly produce the prior time tip/tilt components, $\vartheta^{\mathrm{X}}_{m}$ and $\vartheta^{\mathrm{Y}}_{m}$ rather than calculate these values from the prior time adjusted phase values, $\widetilde{\phi}_{p,m}$.

4. Direct Generation Of Prior Time Tip/Tilt Values

I now offer a final rephrasing of the statement concerning the essence of the RytovProp method, saying:

> If a set of statistically appropriate, randomly selected realization of the P values for $\widetilde{\phi}_{p,0}$, as well as a set of P values for $\widetilde{\ell}_{p,0}$ (both sets for the current time) were available—along with a set of M values for $\vartheta^{\mathrm{X}}_{m}$, as well as a set of M values of $\vartheta^{\mathrm{Y}}_{m}$ (both sets for the prior times), then the value of the Strehl ratio, $\mathcal{S}$, could be calculated using Eq.'s (9).

Compared to the previous statements of the essence of the RytovProp method this version has the advantage of requiring much fewer random variables be generated, making the covariance matrix that will have to be worked with much smaller. But this covariance matrix contains elements for which I have not yet given computational prescriptions.

The covariance matrix will now also contain elements relating the different components of tip/tilt at different (or the same) prior times to each other, elements relating the tip/tilt components at prior times to the current time adjusted phase, and elements relating the tip/tilt components at prior times to the current time adjusted log-amplitude. These elements of the covariance matrix are

$$\mathcal{C}_{\mathrm{XX}}(m;m') = \left\langle\, \vartheta^{\mathrm{X}}_{m}\, \vartheta^{\mathrm{X}}_{m'} \,\right\rangle, \quad \mathcal{C}_{\mathrm{XY}}(m;m') = \left\langle\, \vartheta^{\mathrm{X}}_{m}\, \vartheta^{\mathrm{Y}}_{m'} \,\right\rangle, \quad \text{and} \quad \mathcal{C}_{\mathrm{YY}}(m;m') = \left\langle\, \vartheta^{\mathrm{Y}}_{m}\, \vartheta^{\mathrm{Y}}_{m'} \,\right\rangle, \tag{32}$$

$$\mathcal{C}_{\tilde{\phi}\mathrm{X}}(p,m;m') = \left\langle\, \widetilde{\phi}_{p,m}\, \vartheta^{\mathrm{X}}_{m'} \,\right\rangle, \quad \text{and} \quad \mathcal{C}_{\tilde{\phi}\mathrm{Y}}(p,m;m') = \left\langle\, \widetilde{\phi}_{p,m}\, \vartheta^{\mathrm{Y}}_{m'} \,\right\rangle, \tag{33}$$

and

$$\mathcal{C}_{\tilde{\ell}\mathrm{X}}(p,m;m') = \left\langle\, \widetilde{\ell}_{p,m}\, \vartheta^{\mathrm{X}}_{m'} \,\right\rangle, \quad \text{and} \quad \mathcal{C}_{\tilde{\ell}\mathrm{Y}}(p,m;m') = \left\langle\, \widetilde{\ell}_{p,m}\, \vartheta^{\mathrm{Y}}_{m'} \,\right\rangle. \tag{34}$$

Substituting Eq. (7) into Eq. (32) and carrying out fairly standard simplifications I can write

$$\mathcal{C}_{\mathrm{XX}}(m;m') = k^{-2} \left\langle \sum_{p=1}^{P} \widetilde{x}_{p}\, \widetilde{\phi}_{p,m} \sum_{p'=1}^{P} \widetilde{x}_{p'}\, \widetilde{\phi}_{p',m'} \right\rangle = -k^{-2} \sum_{p,p'=1}^{P} \widetilde{x}_{p}\, \widetilde{x}_{p'}\, \mathcal{D}_{\phi\phi}(p,m;p',m')\,, \tag{35}$$

$$\mathcal{C}_{\mathrm{XY}}(m;m') = k^{-2} \left\langle \sum_{p=1}^{P} \widetilde{x}_{p}\, \widetilde{\phi}_{p,m} \sum_{p'=1}^{P} \widetilde{y}_{p'}\, \widetilde{\phi}_{p',m'} \right\rangle = -k^{-2} \sum_{p,p'=1}^{P} \widetilde{x}_{p}\, \widetilde{y}_{p'}\, \mathcal{D}_{\phi\phi}(p,m;p',m')\,, \tag{36}$$

and

$$\mathcal{C}_{\mathrm{YY}}(m;m') = k^{-2} \left\langle \sum_{p=1}^{P} \widetilde{y}_{p}\, \widetilde{\phi}_{p,m} \sum_{p'=1}^{P} \widetilde{y}_{p'}\, \widetilde{\phi}_{p',m'} \right\rangle = -k^{-2} \sum_{p,p'=1}^{P} \widetilde{y}_{p}\, \widetilde{y}_{p'}\, \mathcal{D}_{\phi\phi}(p,m;p',m')\,, \tag{37}$$

Substituting Eq. (7) into Eq. (33) and carrying out the same sort of simplifications I can write

$$\mathcal{C}_{\tilde{\phi}\mathrm{X}}(p,m;m') = k^{-1} \left\langle \widetilde{\phi}_{p,m} \sum_{p'=1}^{P} \widetilde{x}_{p'}\, \widetilde{\phi}_{p',m'} \right\rangle = k^{-1} \sum_{p'=1}^{P} \widetilde{x}_{p'}\, \Big\{ -\mathcal{D}_{\phi\phi}(p,m;p',m') + \mathcal{D}_{\phi\phi}(0,m;p',m') \Big\}\,, \tag{38}$$

and

$$\mathcal{C}_{\tilde{\phi}\mathrm{Y}}(p,m;m') = k^{-1} \left\langle \widetilde{\phi}_{p,m} \sum_{p'=1}^{P} \widetilde{y}_{p'}\, \widetilde{\phi}_{p',m'} \right\rangle = k^{-1} \sum_{p'=1}^{P} \widetilde{y}_{p'}\, \Big\{ -\mathcal{D}_{\phi\phi}(p,m;p',m') + \mathcal{D}_{\phi\phi}(0,m;p',m') \Big\}\,. \tag{39}$$

And finally, substituting Eq. (7) into Eq. (34) and again carrying out the same sort of simplifications I can write

$$\mathcal{C}_{\tilde{\ell}\mathrm{X}}(p,m;m') = k^{-1}\left\langle \widetilde{\ell}_{p,m} \sum_{p'=1}^{P} \widetilde{x}_{p'}\, \widetilde{\phi}_{p',m'} \right\rangle = k^{-1} \sum_{p'=1}^{P} \widetilde{x}_{p'}\, \mathcal{C}_{\phi\ell}(p',m';p,m)\,, \tag{40}$$

and

$$\mathcal{C}_{\tilde{\ell}\mathrm{Y}}(p,m;m') = k^{-1}\left\langle \widetilde{\ell}_{p,m} \sum_{p'=1}^{P} \widetilde{y}_{p'}\, \widetilde{\phi}_{p',m'} \right\rangle = k^{-1} \sum_{p'=1}^{P} \widetilde{y}_{p'}\, \mathcal{C}_{\phi\ell}(p',m';p,m)\,, \tag{41}$$

With these formulas in hand I proceed to the presentation of sample results.

5. Sample Results: Testing/Validation Of RytovProp

A large set of (split-step, FFT based) wave optics propagation simulations were performed by Barry Foucault, SAIC, simulating propagation from a ground based laser transmitter to a receiver on a satellite in a circular orbit. Results were developed for all combinations of transmitter aperture diameters of $D = 0.1$ m, 0.25 m, and 0.5 m; of laser wave lengths of $\lambda = 0.5$ μm and 1.5 μm, and of tip/tilt tracking servo bandwidths of $f_{3dB} = 0$ Hz, 3 Hz, and 10 Hz. Various engagement conditions—the conditions being different with respect to satellite altitude, to propagation direction zenith angle, and to optical strength of turbulence distributions along the propagation path—were considered.

Cumulative probability distribution results were developed from these wave optics propagation simulation results. For about 6% of the cases only 2,100 frames of data (and Strehl ratio values) were developed; for about another 6% of the cases only about 4,200 frames were developed. For another 21% of the cases no more than 16,800 frames of data were developed. For the remainder of the cases no more than 42,000 frames of data were developed. As a consequence of the limited number of frames of data the cumulative probability distributions are rather unreliable for small probabilities. This is particularly so because there was a high degree of correlation between successive Strehl ratio values—a correlation of 50% at a separation of about ten frames.

Working with all engagement cases and with all system parameters except the $D = 0.5$ m aperture RytovProp results were developed, each with 100,000 statistically independent Strehl ratio values. Cumulative probability distribution results were developed from these RytovProp Strehl ratio results and compared with the cumulative probability distribution results obtained from the wave optics propagation simulations. In Fig. 1 I show a few of these results. In Fig. 2 I show the frame-to-frame (*i.e.* the temporal) correlation of the corresponding wave optics propagation simulation Strehl ratio results. As can be seen there is very good agreement between the wave optics propagation and the RytovProp simulation results, particularly for the higher probabilities, where the wave optics simulation results are less statistically suspect. This agreement is characteristic of that found for all the cases.

6. Acknowledgments

This work was performed under USAF/AFRL Contract No. FA9451-06-C-0348. I would like to thank Dr. Earl Spillar for his encouragement and support of this work. Also I wish to thank Michael Oliker for a great deal of programming support (and for forcing me to pay attention to the occurrence of negative eigen-values and finding a way to avoid their occurrence). Lastly, I would like to thank Barry Foucault for making his wave optics propagation simulation results available to me.

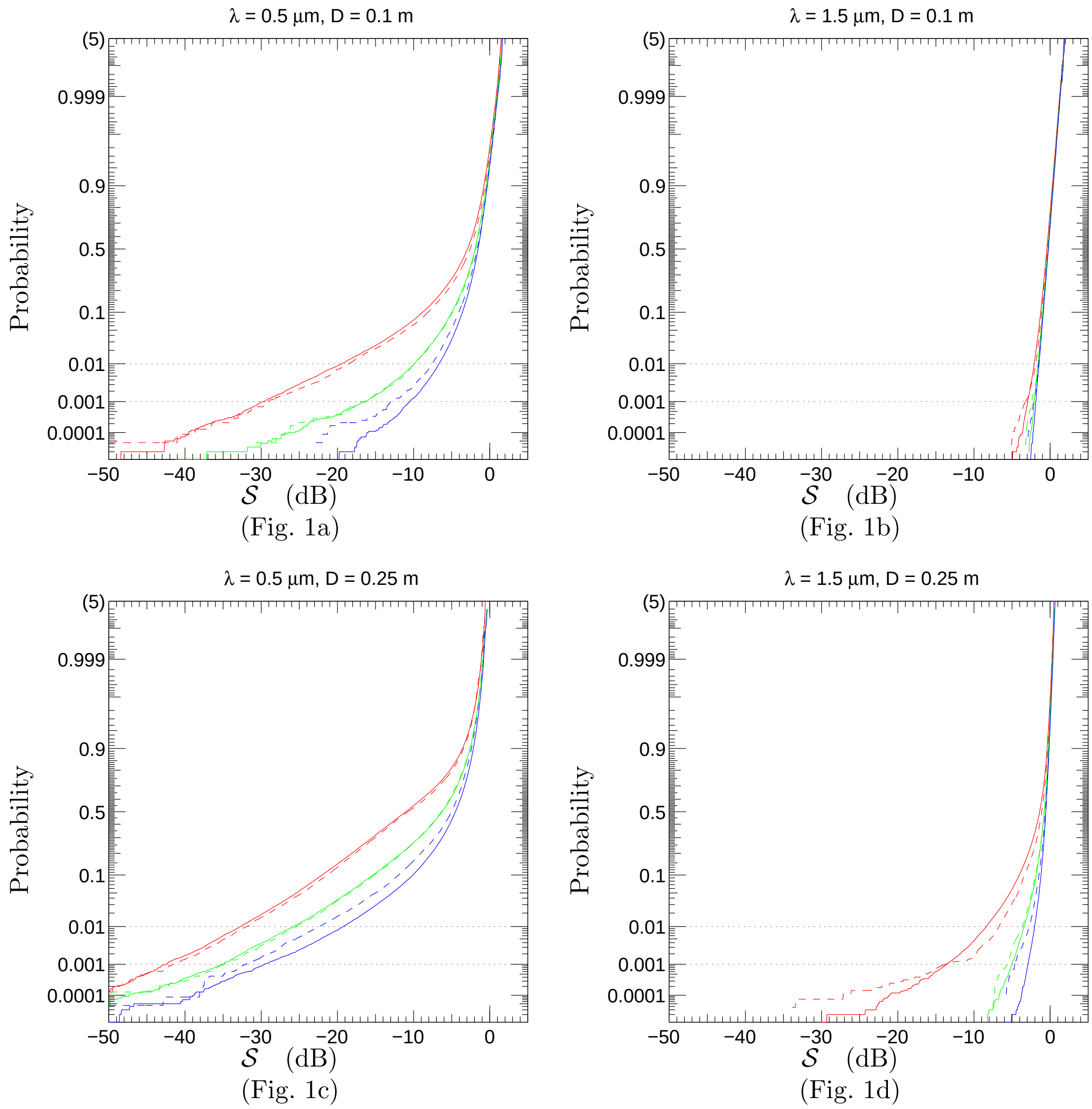
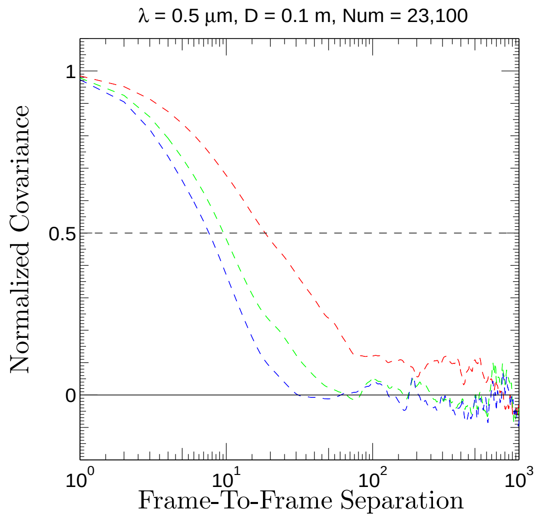
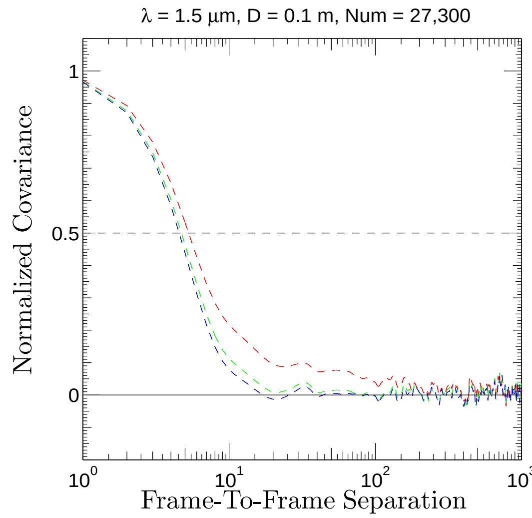


Figure 1. Cumulative Probability Distribution For The First Engagement Scenario

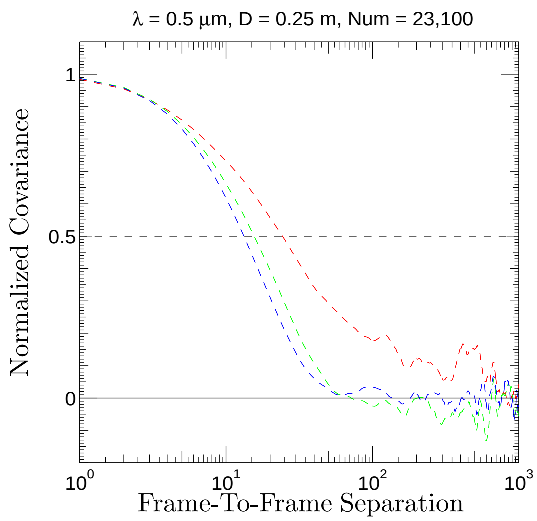
Each of the four plots is for the combinations of aperture diameter, D, and optical wave length, λ, indicated above that plot. The engagement parameters are such that for $\lambda = 0.5\ \mu$m the relevant turbulence parameters have values of $\mathcal{R}\!y = 0.033$ Np^2, $r_0 = 0.103$ m, and $f_{\mathrm{T}} = 16.8$ Hz, while for $\lambda = 1.5\ \mu$m the values are $\mathcal{R}\!y = 0.009$ Np^2, $r_0 = 0.386$ m, and $f_{\mathrm{T}} = 5.6$ Hz. The dashed line curves show the results obtained using the split-step wave optics propagation simulation method. The solid line curves show the results obtained using the RytovProp method. The red line curves are for the case where the tilt tracking servo bandwidth was $f_{3\mathrm{dB}} = 0$ Hz, while the green line and the red line curves are for $f_{3\mathrm{dB}} = 3$ Hz and $f_{3\mathrm{dB}} = 10$ Hz respectively. The horizontal dotted lines indicate the 1% and the 0.1% cumulative probability levels.



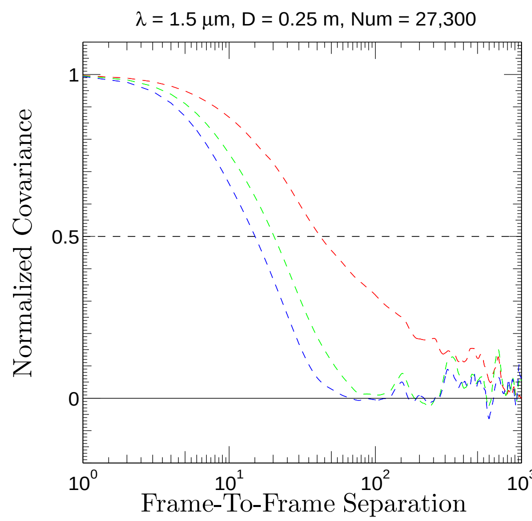
(Fig. 2a)



(Fig. 2b)



(Fig. 2c)



(Fig. 2d)

Figure 2. Normalized Correlation Of Successive Strehl Ratio Values

The three curves in each plot are to be associated with the similarly colored dashed line curves in the corresponding plots of Fig. 1 —the curves based on the wave optics propagation method. The normalized covariance is the covariance divided by the variance. The notation Num above each plot indicates the number of Strehl ratio values used in producing the results shown in Fig. 1. The value of Num divided by the Frame-to-Frame Separation at which the normalized covariance curve crosses the 0.5 level can be taken as an estimate of the number of degrees-of-freedom in the data used in forming the dashed line curves of Fig. 1.