Resolving Rotational Ambiguities for Spin-stabilized Satellites

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Abstract

An analysis is presented of the periodic signal received from a rotating cylindrical regular polygon. All *N* faces of the polygon are assumed to be diffuse reflectors with the same reflectance. The closed-form solution shows that the modulation of the periodic signal falls off as the inverse of the cube of the number of faces.

1. Introduction

The rotation rates of asteroids can be determined by analyzing the periodic frequencies of time-varying photometric signatures from unresolved images of the asteroid. Due to the random nature of the asteroid, in terms of shape and albedo, a unique period can be determined. Because the various sides of an asteroid are different, the smallest repeat cycle of the photometric curve is directly related to the rotation rate of the asteroid.

When this method is applied to spin-stabilized satellites [1], an ambiguity arises. Some spin-stabilized satellites are constructed in the shape of a cylindrical regular polygon. Unless the various sides of the satellite have different albedos or otherwise reflect the light differently, the rotational period of the satellite will be an integer multiple of the smallest period in the signature, where the multiple is equal to the number of identical faces on the satellite. With this ambiguity, one has to know a priori the number of faces on the satellite to measure the rotation rate.

An analysis is presented of the photometric signatures of a diffuse, rotating cylindrical regular polygon and how it relates to the periodicity in the signature. The closed-form solution is limited to the case of zero solar phase angle, i.e. when the illumination comes from behind the observer. The analysis will show that the modulation of the periodic signal decreases as the number of faces on the cylindrical polygon increases. The modulation is defined as the difference divided by the sum of the maximum and minimum values. For a large number of faces, this modulation falls off roughly as the inverse of the cube of the number of faces.

Photometric signatures of satellites in orbit are complicated and will not be adequately represented by a diffuse regular polygon model, but this analysis does provide insight into the multiple-face satellite rotation rate ambiguity.

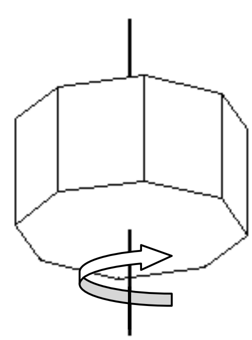

Fig. 1. Rotating diffuse polygon. The model being analyzed is a rotating cylindrically-shaped polygon, as shown above. The cross-section, perpendicular to the axis of rotation of the body, is a regular polygon. Each face of the polygon is assumed to be a diffuse surface of the same constant reflectance, shape and size. The rotation axis, shown by the line above, is perpendicular to the top and bottom surfaces.

2. Diffuse Model

Each face of the rotating polygon is assumed to be a flat diffuse surface of constant reflectance. All faces are assumed to be identical in size, shape and reflectance. The cross-section of the rotating object is a regular polygon with N faces. The normal to each face intersects the axis of rotation and is separated from the normal of the neighboring faces by an angle of $2\pi/N$ radians in either direction. As a result, when the object has undergone a rotation of $2\pi/N$ radians, the new state cannot be distinguished from the previous state.

The angle between the Sun vector and the observer vector is called the solar phase angle. It is represented here by the symbol, Ω. If the angle between the surface normal of the k_{th} face and the observer is represented by the symbol, θ_k, then the angles between the normal and the Sun and observer directions are defined as shown in Figure 2. As is implied in this figure, this paper will restrict its analysis to the case where the axis of rotation is perpendicular to the plane containing the Sun, the observer and the rotating object.

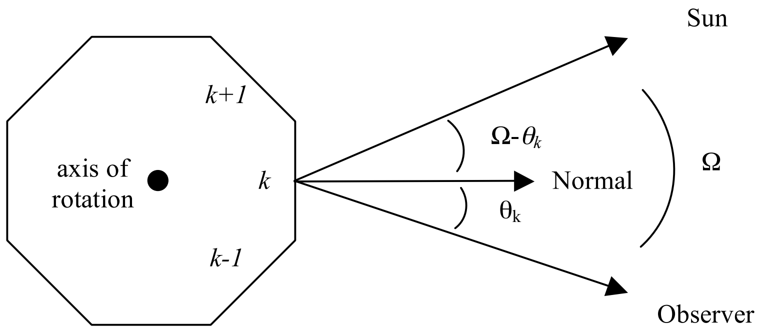

Fig. 2. The solar phase angle is the angle between the vectors from the object to the Sun and to the observer.

For a diffuse surface, the amount of light seen by the observer is proportional to the product of the amount of light impinging on the surface with the fraction of the light seen by the observer. This fraction is equal to the projected area of the surface, as seen by the observer, divided by the true area. This is identically the dot product of the unit normal vector of the surface with the unit observer vector, which is the cosine of the angle between the two vectors.

The amount of light impinging on the surface is also proportional to the projected area of the surface, as seen by the Sun. This is equal to the true area of the surface multiplied by the dot product of the unit normal vector of the surface with the unit Sun vector, or the true area multiplied by the cosine of the angle between the surface normal and the Sun vector.

Combining these two effects together, the amount of light seen by the observer, from all the exposed faces, is proportional to the sum of the reflectances of each face multiplied by the cosine of the angle between each normal and the Sun vector and the cosine of the angle between each normal and the observer vector. Using the angles defined in Figure 2, the photometric signature from the light received by the observer is proportional to:

$$p(t) = aA \sum_{k=-N/2}^{N/2-1} \cos(\theta_k)_+ \cos(\Omega - \theta_k)_+ \tag{1}$$

where the angles θ_k and Ω, although not shown explicitly, are functions of time. The constant, a, is the albedo or the reflectance of each surface and the constant, A, is the true area of each surface. The plus sign subscript notation on the cosine functions indicates that the cosine function contributes to the signal only when the cosine value is positive, otherwise it has a zero contribution. The first cosine function will be negative when that face is out of view of the observer, while the second cosine function will be negative when that face is in shadow and is not illuminated.

Since there are N faces around 2π radians, the normals for adjacent faces are separated by a fixed angle of $2\pi/N$. The N faces are numbered $-N/2$ to $N/2-1$, where the normal of the 0^{th} face is pointing directly at the observer at time $t=0$. With a constant rotation rate of ω radians/second, the observer angle of the k^{th} face is given by:

$$\theta_k(t) = \omega t + k\frac{2\pi}{N}, \qquad \text{for } -\frac{N}{2} \le k \le \frac{N}{2} - 1 \tag{2}$$

With N identical faces, the photometric signature has a periodicity that corresponds to the time it takes for the polygon to rotate by $2\pi/N$ radians. In other words, there are N identical periodic cycles in one rotation period of the polygon. Each periodic cycle extends over $2\pi/N$ radians. If the rotation angle, ωt, is designated by, α, then using Equations 1 and 2, the photometric signal over one periodic cycle is given by:

$$p(\alpha) = aA\sum_{k=-N/2}^{N/2-1} \cos\left(\alpha + k\frac{2\pi}{N}\right)_+ \cos\left(\Omega - \alpha - k\frac{2\pi}{N}\right)_+ \qquad \text{for } 0 \le \alpha \le \frac{2\pi}{N} \tag{3}$$

As the number of faces, N, increases, the polygon-shaped object becomes a cylinder. The cross-section of the cylinder is the circle that encloses the polygon, as seen in Figure 3. From this figure, the width of one face of the polygon, W, can be written in terms of the radius of the cylinder, R, and the number of faces.

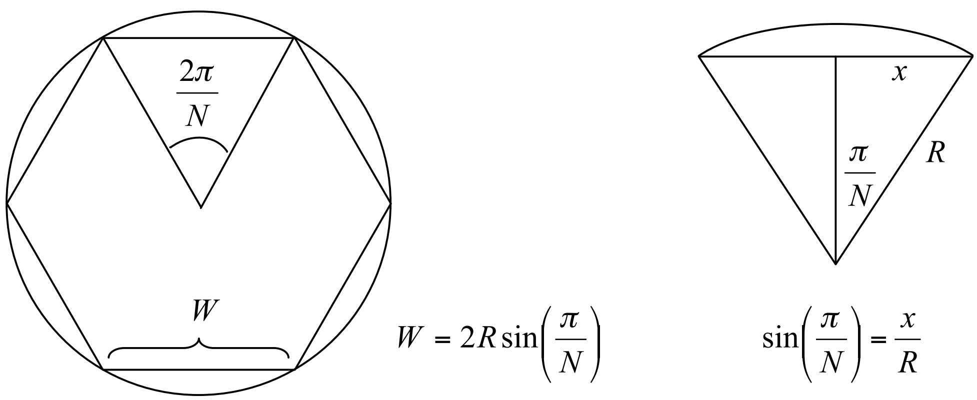

Fig. 3. For a large number of faces, N, the polygon approaches the enclosing cylinder.

If the height of the cylinder is H, then the area of one face of the polygon, A, is equal to WH. The photometric signal for a diffuse cylinder has been shown [2] to be:

$$P = \frac{1}{2}aRH\left[(\pi - \Omega)\cos(\Omega) + \sin(\Omega)\right] \tag{4}$$

By dividing the photometric signature by the signature of the cylinder, a redefined photometric signature is:

$$p(\alpha) = \frac{4\sin(\pi/N)}{(\pi - \Omega)\cos(\Omega) + \sin(\Omega)}\sum_{k=-N/2}^{N/2-1} \cos\left(\alpha + k\frac{2\pi}{N}\right)_+ \cos\left(\Omega - \alpha - k\frac{2\pi}{N}\right)_+ \tag{5}$$

In this paper, we will only consider the case of a zero solar phase angle, i.e. the Sun is directly behind the observer. With that condition ($\Omega=0$), the final expression for the relative photometric signature becomes:

$$p(\alpha) = \frac{4}{\pi}\sin\left(\frac{\pi}{N}\right)\sum_{k=-N/2}^{N/2-1} \cos^2\left(\alpha + k\frac{2\pi}{N}\right)_+ \qquad \text{for } 0 \le \alpha \le \frac{2\pi}{N} \tag{6}$$

Simulation

A Matlab simulation of the periodic signature of the rotating polygon is shown below in Figure 4. Equation 6 is evaluated in the simulation for several different values of the number of faces of the polygon. The results of the simulation are shown in Figure 5.

The simulation evaluates the signature over a range of the rotation angle, α, of 2π radians, or one complete rotation of the polygon. For a polygon of N faces, there will be N cycles of the periodic signature over this range. In the simulation, the signature is evaluated at a thousand points, which is sufficient accuracy to sample the signatures.

The modulation of the periodic signature is defined as the difference of the maximum and minimum values divided by the sum of the two values. From Figure 5, it can be seen that the modulation is identically zero for all even numbers of faces greater than 2. For odd numbers of faces, the modulation is not zero, but if falls off rapidly as the number of faces increases.

```matlab
% rotating_polygon.m

phase_angle = 0*pi/180;
N = 20;

f = figure;

for nfaces=2:N

% Single facet
albedo_area = 4*sin(pi/nfaces)/((pi-phase_angle)*cos(phase_angle)+sin(phase_angle));
angle_facet = 2*pi/nfaces;

% Do 1000 samples per cycle
normal = repmat(2*pi*[0:(nfaces-1)]'/nfaces,[1 1001])+repmat([0:1000]*2*pi/1000,[nfaces 1]);
observer = cos(normal);
sun = cos(normal).*cos(phase_angle)+sin(normal).*sin(phase_angle);
diffuse = sum(albedo_area*observer.*(observer > 0).*sun.*(sun > 0));

% Plot the result
figure;
plot(2*[0:1000]/1000,diffuse,'r','linewidth',2);
title(sprintf('Faces=%g, Phase angle=%g',nfaces,phase_angle));
axis([0 2 0 1.3]);

% Calculate the modulation
maxv = max(diffuse(:));
minv = min(diffuse(:));
modulation(nfaces) = (maxv-minv)/(maxv+minv);

end

% Plot the modulation
figure(f);
plot(2:N,modulation(2:N).^0.5,'r',2:N,modulation(2:N).^0.5,'ob','linewidth',2);
title('Modulation of Polygon');
xlabel('Number of Faces');
ylabel('Modulation Square Root');
```

Fig. 4. Matlab implementation of Equation 6.

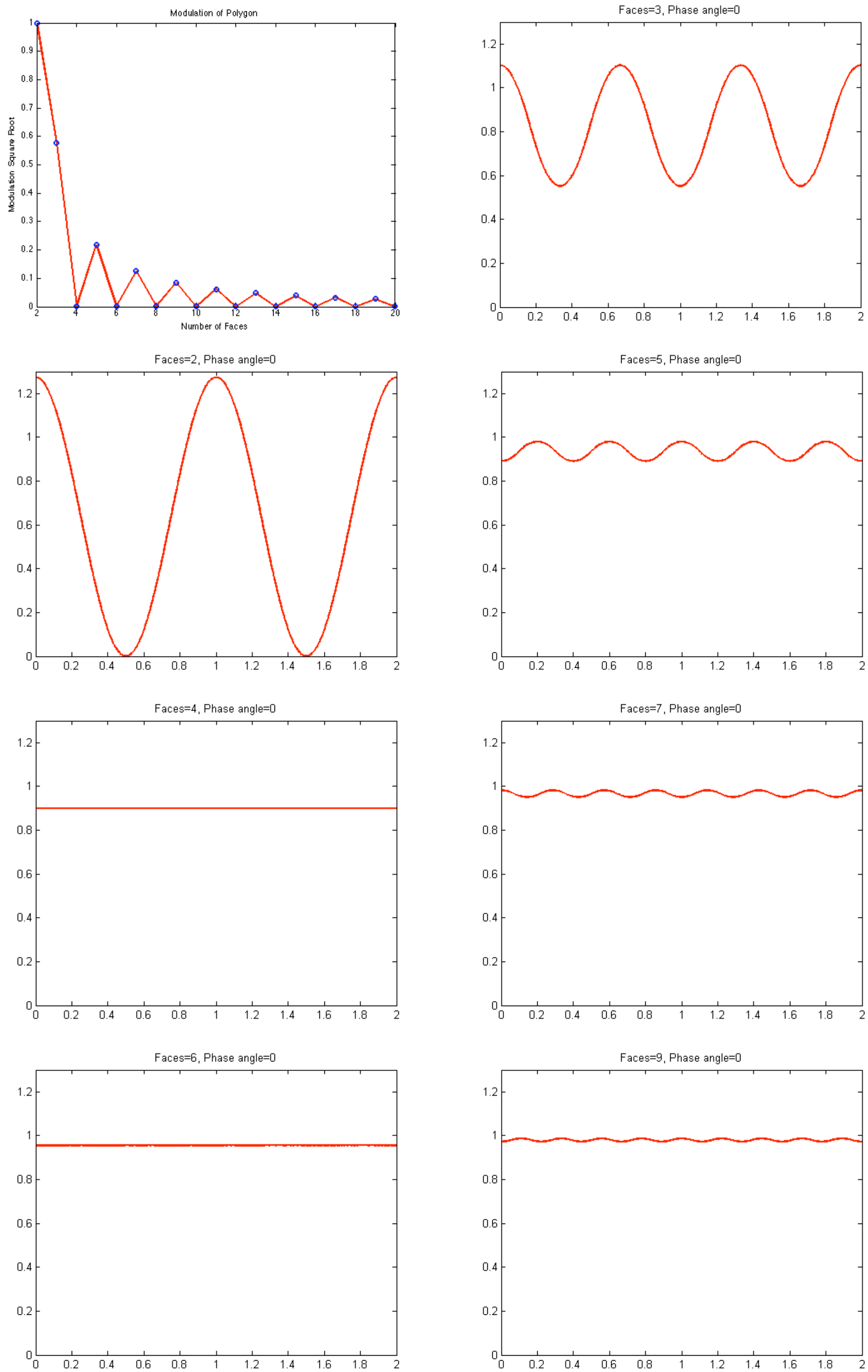

Fig. 5. Simulation results. The signatures for an even number of faces are shown on the left and an odd number of faces on the right. Starting at $N=4$ faces, the even numbered cases identically have no periodicity. The odd numbered cases, on the other hand, show a decreasing modulation as the number of faces increases. A plot of the modulation of the signatures, as a function of the number of faces, is shown on the top of the left column.

3. Theoretical Solution

In the simulation, the cosine functions, or dot products, can be tested and for positivity and be included only if they are greater than zero. That technique does not work for the theoretical solution. A different method is needed to include only the positive contributions of the cosine functions. Since the cosine is negative only when its argument is outside of the range *±π/2*, the limits on the sum can be modified to exclude those terms.

A negative cosine is the same as saying that a face no longer contributes when the angle with respect to the Sun and observer vectors is *±π/2* or greater. From Fig. 8, it can be seen that, at *t=0*, a face contributes when

$$|k|\frac{2\pi}{N} \le \frac{\pi}{2} \quad \text{or} \quad |k| \le \frac{N}{4} \tag{7}$$

This is roughly half the number of faces for a zero phase angle. The value of k will be the integer whose absolute value is less than or equal to *N/4*.

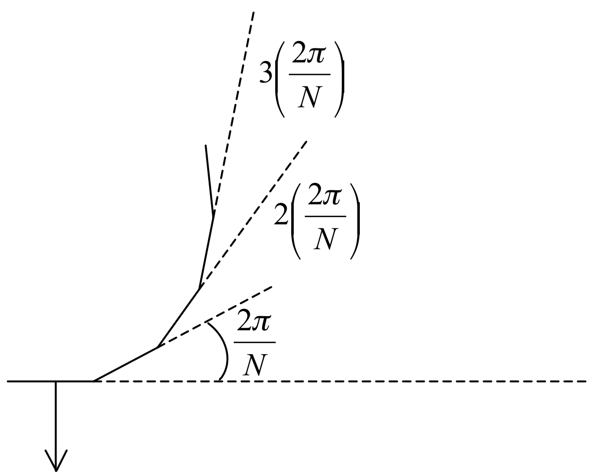

Fig. 6. Only those faces whose angles with respect to the Sun/Observer direction are less than *π/2* will contribute.

The exact number of faces included in the limits on the summation in Equation 6 depends on two parameters, the number of faces on the polygon, N, and the number of faces on one side (or one half) of the polygon, *N/2*.

For an even number of faces, N, the parameter can be expressed as a multiple of 2:

$$N = 2q \tag{8}$$

where q is an integer. The parameter q is the second parameter, i.e. it is roughly the number of faces on one side or one half of the polygon. If the polygon has an even number of faces, since one face is pointed at the Sun and the observer, as shown in Figure 6, then there is also one face on the other side facing away from the observer. This means that there is one face is pointing to the observer and there are *q-1* faces pointing to one side.

For an odd number of faces, N, the parameter can be expressed as a multiple of 2 plus one:

$$N = 2q + 1 \tag{9}$$

where q is an integer. Since the polygon has an odd number of sides, one face is pointed at the Sun and the observer and there are q faces pointing to one side or the other.

For each of these two cases, there are two more cases, i.e. when the number of faces on one side, q, is odd or even. If the number of faces pointing to one side is odd, then there is a face pointing directly to the side. That face does not start out illuminated or visible. On one side of the polygon, the face pointing to the side rotates out of view and on the other side of the polygon, it rotates into view for one periodic cycle.

When the number of faces on one side, q, is even, there is no face pointing directly to the side at $t=0$. That means that a vertex is pointing directly to the side with two faces pointing half of a rotation cycle, or π/N radians, either towards the observer or away from the observer. As a result, halfway through the rotation cycle a face on one side will leave the summation and on the other side a face will enter the summation.

If we make the assumption that the polygon in Figure 6 is rotating counterclockwise and the angles of the normals are negative on the left side of the polygon and positive on the right side of the polygon, then the limits on the summation in Equation 6 are:

N even, q odd

$$0 \le \alpha \le \frac{\pi}{N} \qquad -\frac{N-2}{4} \le k \le \frac{N-2}{4} \tag{10}$$

$$\frac{\pi}{N} \le \alpha \le \frac{2\pi}{N} \qquad -\frac{N-2}{4} - 1 \le k \le \frac{N-2}{4} - 1 \tag{11}$$

N even, q even

$$0 \le \alpha \le \frac{2\pi}{N} \qquad -\frac{N-4}{4} - 1 \le k \le \frac{N-4}{4} \tag{12}$$

N odd, q odd

$$0 \le \alpha \le \frac{1}{2}\frac{\pi}{N} \qquad -\frac{N-3}{4} \le k \le \frac{N-3}{4} \tag{13}$$

$$\frac{1}{2}\frac{\pi}{N} \le \alpha \le \frac{3}{2}\frac{\pi}{N} \qquad -\frac{N-3}{4} - 1 \le k \le \frac{N-3}{4} \tag{14}$$

$$\frac{3}{2}\frac{\pi}{N} \le \alpha \le \frac{2\pi}{N} \qquad -\frac{N-3}{4} - 1 \le k \le \frac{N-3}{4} - 1 \tag{15}$$

N odd, q even

$$0 \le \alpha \le \frac{1}{2}\frac{\pi}{N} \qquad -\frac{N-1}{4} \le k \le \frac{N-1}{4} \tag{16}$$

$$\frac{1}{2}\frac{\pi}{N} \le \alpha \le \frac{3}{2}\frac{\pi}{N} \qquad -\frac{N-1}{4} \le k \le \frac{N-1}{4} - 1 \tag{17}$$

$$\frac{3}{2}\frac{\pi}{N} \le \alpha \le \frac{2\pi}{N} \qquad -\frac{N-1}{4} - 1 \le k \le \frac{N-1}{4} - 1 \tag{18}$$

Using these limits on the summation, a general expression for the relative photometric signal can be written as:

$$p(\alpha) = \frac{4}{\pi}\sin\left(\frac{\pi}{N}\right)\sum_{k=-Q-A}^{Q-B}\cos^2\left(\alpha + k\frac{2\pi}{N}\right) \tag{19}$$

where the constants, Q, A and B, are determined from the relations in Equations 10-18. The number of faces on one side of the polygon that are illuminated at time $t=0$, represented by the constant Q, varies from $(N-1)/4$ to $(N-4)/4$. The constants A and B are the number of faces that appear or disappear as the polygon rotates, and equal to 0 or 1.

N	q	± Q	A	B	rotation cycle (radians)
even	odd	(N-2)/4	0	0	$0 \le \alpha \le \pi/N$
	odd	(N-2)/4	1	1	$\pi/N \le \alpha \le 2\pi/N$
even	even	(N-4)/4	1	0	$0 \le \alpha \le 2\pi/N$
odd	odd	(N-3)/4	0	0	$0 \le \alpha \le \pi/2N$
	odd	(N-3)/4	1	0	$\pi/2N \le \alpha \le 3\pi/2N$
	odd	(N-3)/4	1	1	$3\pi/2N \le \alpha \le 2\pi/N$
odd	even	(N-1)/4	0	0	$0 \le \alpha \le \pi/2N$
	even	(N-1)/4	0	1	$\pi/2N \le \alpha \le 3\pi/2N$
	even	(N-1)/4	1	1	$3\pi/2N \le \alpha \le 2\pi/N$

Table 1. For the four cases of N and q both odd and even, the limits *(Q)* represent the initial number of illuminated faces. In different regions of the rotation cycle, single faces enter *(A)* and leave *(B)* the field of view.

Using the fact that A and B only add or subtract one additional term in the summation, the expression for the relative photometric signal can be written as the sum of three terms, where the A term adds a contribution from a face appearing on the left and the B term subtracts a contribution from a face disappearing on the right:

$$p(\alpha) = \frac{4}{\pi}\sin\left(\frac{\pi}{N}\right)\left[S(\alpha) + A\cos^2\left(\alpha - (Q+1)\frac{2\pi}{N}\right) - B\cos^2\left(\alpha + Q\frac{2\pi}{N}\right)\right] \tag{20}$$

where

$$S(\alpha) = \sum_{-Q}^{Q}\cos^2\left(\alpha + k\frac{2\pi}{N}\right) \tag{21}$$

Using the expression for the cosine of the double angle:

$$S(\alpha) = \sum_{-Q}^{Q}\left[\frac{1}{2} + \frac{1}{2}\cos\left(2\alpha + k\frac{4\pi}{N}\right)\right] \tag{22}$$

and the expression for the cosine of the sum of two angles:

$$S(\alpha) = Q + \frac{1}{2} + \frac{1}{2}\cos(2\alpha)\sum_{-Q}^{Q}\cos\left(k\frac{4\pi}{N}\right) - \frac{1}{2}\sin(2\alpha)\sum_{-Q}^{Q}\sin\left(k\frac{4\pi}{N}\right) \tag{23}$$

The sum over *sin(k)* is symmetric around zero and the positive and negative terms exactly cancel.

The finite sum over cosine terms has been shown [3, 4] to be:

$$\sum_{k=-Q}^{Q}\cos\left(k\frac{4\pi}{N}\right)=2\cos\left(Q\frac{2\pi}{N}\right)\frac{\sin\left((Q+1)\frac{2\pi}{N}\right)}{\sin\left(\frac{2\pi}{N}\right)}-1 \tag{24}$$

Substituting this into Equation 23 yields:

$$S(\alpha)=Q+\frac{1}{2}-\frac{1}{2}\cos(2\alpha)+\cos(2\alpha)\cos\left(Q\frac{2\pi}{N}\right)\frac{\sin\left((Q+1)\frac{2\pi}{N}\right)}{\sin\left(\frac{2\pi}{N}\right)} \tag{25}$$

For $Q = (N-i)/4$, where $i = [1:4]$, the summation is:

$$S(\alpha)=\frac{N}{4}+\frac{2-i}{4}-\frac{1}{2}\cos(2\alpha)+\cos(2\alpha)\frac{\sin\left(i\frac{\pi}{2N}\right)\cos\left((4-i)\frac{\pi}{2N}\right)}{\sin\frac{2\pi}{N}} \tag{26}$$

The additional terms, modified by A and B in Equation 20, can be further expanded with the same relationship:

$$p(\alpha)=\frac{4}{\pi}\sin\left(\frac{\pi}{N}\right)\left[S(\alpha)+A\cos^2\left(\alpha-\left(\frac{N-i}{4}+1\right)\frac{2\pi}{N}\right)-B\cos^2\left(\alpha+\left(\frac{N-i}{4}\right)\frac{2\pi}{N}\right)\right] \tag{27}$$

Eliminating the $\pi/2$ shifts and using the double angle cosine relationship with sine squared yields:

$$p(\alpha)=\frac{4}{\pi}\sin\left(\frac{\pi}{N}\right)\left[S(\alpha)+\left(\frac{A-B}{2}\right)-\frac{A}{2}\cos\left(2\alpha-(i-4)\frac{\pi}{N}\right)+\frac{B}{2}\cos\left(2\alpha-i\frac{\pi}{N}\right)\right] \tag{28}$$

With the reduced forms in Equations 26 and 28, and using Table 1, it is possible to determine the closed form solution to the periodic signal for all four cases of the types of faces on the polygon. Despite the simplicity so far, it will take further reduction to achieve the final results.

The easiest cases to consider are when the number of faces, N, is even. When q is odd, there are two cases, one with no extra cosine terms and one with both extra terms. Because for this case, $i = 2$, the two cosine terms exactly cancel, so they do not contribute. As can be seen in Equation 26, when $i = 2$, the terms that are a function of α exactly cancel and the remaining contribution in Equation 28 is a constant.

When $i = 4$, the second case for N even, the terms that are a function of α in Equation 26 do not cancel, but they are equal and opposite to the A term which contributes to this case. The result is the same constant value for the periodic signature. As a result, when the number of faces, N, is even, the periodic signature is a constant of value:

$$p(\alpha)=\frac{N}{\pi}\sin\left(\frac{\pi}{N}\right) \tag{29}$$

In the limit as $N \to \infty$, the relative periodic signature approaches unity, i.e. the value returned by a cylinder. This is an interesting result to consider. If the number of faces of the polygon, N, is even, then the return signature is a constant, i.e. there is no periodic signature to analyze. This is not the case for when N is odd. This latter case is also more difficult to compute.

For the cases when N is odd, the value of $i = [1, 3]$. In addition, there are three different ranges in the rotation cycle and the constants A and B are either both equal to 0, or both equal to 1, or one is 0 and the other is 1.

The case of $A = B = 0$ is the easiest to consider. The only contribution to the relative periodic signature is from the main summation in Equation 26. The results of that equation for $i = 1$ or $i = 3$ are symmetric and of opposite sign, therefore, in this analysis we will only demonstrate how to calculate the case for $i = 1$. Substituting this value for i in Equation 26 yields:

$$S(\alpha) = \frac{N}{4} + \frac{1}{4} - \frac{1}{2}\cos(2\alpha) + \cos(2\alpha)\frac{\sin\left(\frac{\pi}{2N}\right)\cos\left(\frac{3\pi}{2N}\right)}{\sin\frac{2\pi}{N}} \tag{30}$$

At first glance the terms multiplying the $\cos(2\alpha)$ appear difficult to reduce, but if each term is re-written as a difference from the angle π/N, then the terms greatly simplify. The multiplier of the second $\cos(2\alpha)$ term is:

$$\frac{\sin\left(\frac{\pi}{2N}\right)\cos\left(\frac{3\pi}{2N}\right)}{\sin\frac{2\pi}{N}} = \frac{\sin\left(\frac{\pi}{N} - \frac{\pi}{2N}\right)\cos\left(\frac{\pi}{N} + \frac{\pi}{2N}\right)}{2\sin\frac{\pi}{N}\cos\frac{\pi}{N}} \tag{31}$$

Expanding these terms with the sums of angles for sines and cosines and combining terms yields:

$$\frac{\sin\left(\frac{\pi}{2N}\right)\cos\left(\frac{3\pi}{2N}\right)}{\sin\frac{2\pi}{N}} = \frac{\cos\left(\frac{\pi}{N}\right) - \frac{1}{2}}{2\cos\frac{\pi}{N}} \tag{32}$$

Substituting this into Equation 26, with $i = 1$, gives for the first range, a final result of:

$$p(\alpha) = \frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N+1) - \frac{\cos(2\alpha)}{\cos\left(\frac{\pi}{N}\right)}\right] \quad \text{for} 0 \le \alpha \le \frac{1}{2}\frac{\pi}{N} \tag{33}$$

For the second range, when N is odd and q is even, the B cosine term is added to the above answer. Substituting $A = 0$, $B = 1$ and $i = 1$ into Equations 28 and adding it to Equation 33 yields:

$$p(\alpha) = \frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N+1) - \frac{\cos(2\alpha)}{\cos\left(\frac{\pi}{N}\right)} - 2 + 2\cos\left(2\alpha - \frac{\pi}{N}\right)\right] \quad \text{for} \frac{1}{2}\frac{\pi}{N} \le \alpha \le \frac{3}{2}\frac{\pi}{N} \tag{34}$$

or

$$p(\alpha)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N-1)-\frac{\cos(2\alpha)-2\cos\left(2\alpha-\frac{\pi}{N}\right)}{\cos\left(\frac{\pi}{N}\right)}+\right] \quad \text{for}\frac{1}{2}\frac{\pi}{N}\le\alpha\le\frac{3}{2}\frac{\pi}{N} \tag{35}$$

Re-writing the *cos(2α-π/N)* as *cos(2α-2π/N+π/N)* and expanding and combining terms results in the simple expression:

$$p(\alpha)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N-1)+\frac{\cos\left(2\alpha-\frac{2\pi}{N}\right)}{\cos\left(\frac{\pi}{N}\right)}+\right] \quad \text{for}\frac{1}{2}\frac{\pi}{N}\le\alpha\le\frac{3}{2}\frac{\pi}{N} \tag{36}$$

Similarly, adding both the *A* and *B* terms for the third rotation cycle range yields a periodic signature of:

$$p(\alpha)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N+1)-\frac{\cos\left(2\alpha-\frac{4\pi}{N}\right)}{\cos\left(\frac{\pi}{N}\right)}+\right] \quad \text{for}\frac{3}{2}\frac{\pi}{N}\le\alpha\le\frac{2\pi}{N} \tag{37}$$

These results show that when N is odd, there is a periodic signature that changes within the rotation cycle. The complete summary of solutions for all cases is:

N even:

$$p(\alpha)=\frac{N}{\pi}\sin\left(\frac{\pi}{N}\right) \qquad \text{for } 0\le\alpha\le\frac{2\pi}{N} \tag{38}$$

N odd, q even:

$$p(\alpha)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N+1)-\frac{\cos(2\alpha)}{\cos\left(\frac{\pi}{N}\right)}\right] \qquad \text{for } 0\le\alpha\le\frac{1}{2}\frac{\pi}{N} \tag{39}$$

$$p(\alpha)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N-1)+\frac{\cos\left(2\alpha-\frac{2\pi}{N}\right)}{\cos\left(\frac{\pi}{N}\right)}\right] \quad \text{for } \frac{1}{2}\frac{\pi}{N}\le\alpha\le\frac{3}{2}\frac{\pi}{N} \tag{40}$$

$$p(\alpha)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N+1)-\frac{\cos\left(2\alpha-\frac{4\pi}{N}\right)}{\cos\left(\frac{\pi}{N}\right)}\right] \quad \text{for } \frac{3}{2}\frac{\pi}{N}\le\alpha\le\frac{2\pi}{N} \tag{41}$$

and for N odd, q odd

$$p(\alpha)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N-1)+\frac{\cos(2\alpha)}{\cos\left(\frac{\pi}{N}\right)}\right] \qquad \text{for } 0 \le \alpha \le \frac{1}{2}\frac{\pi}{N} \tag{42}$$

$$p(\alpha)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N+1)-\frac{\cos\left(2\alpha-\frac{2\pi}{N}\right)}{\cos\left(\frac{\pi}{N}\right)}\right] \quad \text{for } \frac{1}{2}\frac{\pi}{N} \le \alpha \le \frac{3}{2}\frac{\pi}{N} \tag{43}$$

$$p(\alpha)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[(N-1)+\frac{\cos\left(2\alpha-\frac{4\pi}{N}\right)}{\cos\left(\frac{\pi}{N}\right)}\right] \quad \text{for } \frac{3}{2}\frac{\pi}{N} \le \alpha \le \frac{2\pi}{N} \tag{44}$$

4. Modulation

The modulation of the signature is defined [5] as the ratio of the difference to the sum of the maximum and minimum values in a periodic signal. For the relative periodic signature of the rotating polygon, the modulation is:

$$\text{modulation} = \frac{P_{\max} - P_{\min}}{P_{\max} + P_{\min}} \tag{45}$$

The modulation varies between 0 and 1. It can also be expressed as a percentage. When the minimum value is zero, then the modulation is 1, or 100%. This definition of modulation is independent of scale factors. If the signal is multiplied by a constant factor, then both the minimum and maximum values are multiplied by the same factor and the modulation is unchanged.

The maxima and minima are found by differentiating the closed form solutions in Equations 39-44 and finding the value of α for which the derivatives are zero. By inspection, the only terms that are differentiable are the cosine terms. The derivative of the cosine is the sine and the sine is zero when its argument is zero. That means that the minimum and maximum are located at $\alpha = 0$ and $\alpha = \pi/N$. The signs of the second derivatives at those values indicate which is the maximum and which is the minimum. Evaluating the expressions at those values gives the following values for the minimum and maximum values:

$$P_{\min} = p_1(0)=\frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[N+1-\frac{1}{\cos\left(\frac{\pi}{N}\right)}\right] \tag{46}$$

$$P_{\max} = p_2\left(\frac{\pi}{N}\right) = \frac{1}{\pi}\sin\left(\frac{\pi}{N}\right)\left[N - 1 + \frac{1}{\cos\left(\frac{\pi}{N}\right)}\right] \tag{47}$$

Substituting these values into Equation 45 and combining terms yields the following simple expression:

$$\text{modulation} = \frac{1 - \cos\left(\frac{\pi}{N}\right)}{N\cos\left(\frac{\pi}{N}\right)} \tag{48}$$

For the interesting case when N is large, the cosine can be approximated as:

$$\cos\left(\frac{\pi}{N}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{N}\right)^2 \tag{49}$$

The modulation, under these conditions, tends towards:

$$\text{modulation} \approx \frac{\pi^2/2}{N^3} \tag{50}$$

Figure 7 shows a comparison of the simulation with the theory in Equation 48 and the approximation in Equation 50. After only a few faces, the approximation is quite good.

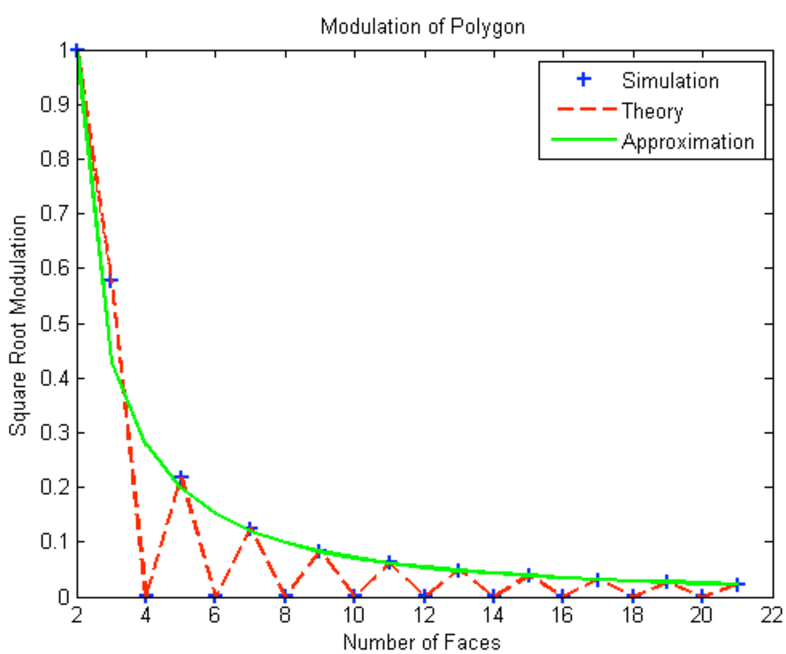

Figure 7. Comparison of the theoretical modulation with simulation.

5. Conclusions

The modulation of the periodic signature of a cylindrically-shaped, rotating polygon has been shown to be dependent on the number of faces of the polygon. Only the specific case of a zero solar phase angle, i.e. when the Sun is located behind the observer, was calculated in this paper. For that specific case, polygons with an even number of faces had no modulation at all. Only polygon with an odd number of faces exhibited periodic signals in the photometric signature. For these signatures, the modulation varied proportionally to the inverse third power of the number of faces. These results lend support to the concept that the number of faces of a rotating polygon might be determinable from the modulation of the periodic signature.

6. References

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4. The solution given in reference [4] for Equation 196 is wrong. The correct solution was determined from Mathematica and confirmed by recursion.
5. W. H. Steel, Interferometry, Cambridge University Press, Cambridge, United Kingdom, p. 44 (1967).