Closely-spaced objects and mathematical groups combined with a robust observational method

Paul LeVan
Air Force Research Laboratory
Space Vehicles Directorate
Kirtland Air Force Base, NM 87117-5776

ABSTRACT

Geosynchronous objects appear as unresolved blurs when observed with even the largest ground-based telescopes. Due to the lack of any spatial detail, two or more objects appearing at similar brightness levels within the spectral bandpass they are observed are difficult to distinguish. Observing a pattern of such objects from one time epoch to another leads to deficiencies in associating individual objects before and after the observation. Only by invoking an additional discrimination measure can the individual objects be kept separated between separate observations. A methodology for accomplishing this is described in this paper.

1. INTRODUCTION

Closely spaced objects present a particular challenge to SSA. For the discussions that follow, by closely spaced is meant a grouping of two or more objects that are distant and individually unresolved, but are separated sufficiently to be resolved from one another. These groupings could be in a congested area of the geosynchronous belt, for example. The problem for SSA is that observations are not continuous, but rather episodic. When a grouping of two or more objects is re-observed, the relative positions may have changed. In the case of geosynchronous objects with similar brightness levels and no light curve signature, the identity of the individual objects can be lost between the observations.

Although problematic for SSA, the above scenario is an interesting application of the separate areas of mathematics as well as astronomical observation techniques. The math area that describes this problem well is Group Theory – and specifically the permutation groups. The latter is easily understood for the grouping of two closely-space objects: the permutation group is P2, consisting of the identity (no change in relative positions) and a transposition (or exchange of the two objects), and is a group of order 2. For 3 objects, P3 has six permutations (identity, three odd, and two even, cyclical permutations), and is obviously of much increased complexity relative to P2. It should be understood that a practical application of the theory is possible for all observed patterns of objects – one only needs to order them in a consistent (and unique) way; e.g., by increasing longitude or, more robustly, by distance metric involving both latitude and longitude.

2. RATIOS OF OBSERVED BJECT SIGNALS

The implications of group theory for this SSA application become more profound when coupled with the discrete number of ratios corresponding to object signals. Here, we envision a 'signature' for each object that is propagated through the atmosphere and measured by a sensor with a multiplicative calibration equation. This class of observation applies to systems with linear response and includes spectrometers (which have a separate multiplicative correction for each wavelength channel) as well as common broadband imagers. The atmospheric correction is also multiplicative (and distinct for the different wavebands), so the corrected SSA 'signature' looks like

$$SS^i_j = (1/T_j\, R_j)\, Sig^i_j,$$

… where j is the channel (e.g., waveband) and i is the observed object. The factors T_j and R_j are atmospheric transmission and instrumental response. (We focus on ground-based here, but with atmospheric transmission ignored, results should be applicable to space-based observations as well.)

In an effort to facilitate the speed of understanding these CSO signatures and performing the identify function, it makes sense to avoid the atmospheric correction process (which can have relatively short-term variability and adds complexity to data reduction) and instead deal with ratios of signals between pairs of objects. These ratios are independent of both atmospheric transmission variations and instrumental response. For the 2-object cluster, there's only one ratio and its inverse; these we denote by R12 (and R21), the ratio of the signal from 'Object 1' to the signal from 'Object 2' (and vice-versa). The S2 permutation group applied to this ratio gives R12 (the identity, with no exchange of objects) and R21 (= 1/R12, the simple exchange of the two objects). Thus, in the case of two objects having a unique ratio of 'signatures' (e.g., spectral), we can identify the individual objects even though they might be of similar brightness and have surreptitiously changed positions between observations with a standard "metric" sensor.

The 'signature ratios' from three objects also follow from the S3 permutation group. The allowed ratios are: {R12, R13, R23, and their inverses R21, R31, and R32}. Here again we assume that although the brightnesses of the objects observed with a standard metric sensor for SSA are similar to the point of not allowing identification, the SSA 'signature ratios' are all distinct – derived from spectral signals in a way described below.

Below are the results of the S3 permutation group applied to the first set of ratios of the first observations {Rij}, and those newly observed at a later epoch, {Rij'}.

Table 1. Transformation of observed ratios under all possible permutations. (Rij' denotes second set of observations.)

Permutation	**R12'**	**R13'**	**R23'**
Identity	R12	R13	R23
(1,2)	R21 = 1/R12	R23	R13
(1,3)	R32 = 1/R23	R31 = 1/R13	R21 = 1/R12
(2,3)	R13	R12	R32 = 1/R23
(231) = (2,3)(1,2)	R23	R21 = 1/R12	R31 = 1/R13
(312) = (1,3)(1,2)	R31 = 1/R13	R32 = 1/R23	R12

This table is fully describes all the possibilities. Each newly-observed ratio is directly related to previously-observed ratios.

There is a graphical display of these permutations for 3 CSOs with which an 'operator' would find it rather easy to interpret the observed ratios. It deals (simply) with two ratios (since the third can be obtained from the other two – we're limiting ourselves to relative ratios, which implies that there are only two variable "SSA signatures", and two unique ratios).

For the first set of observations (involving both metric positions and the "SSA signature" signals for each object), the 'key' ratios R13 and R23 are plotted. In this example, Object 3 is the 'faintest' in terms of the SSA signature, and R23 and R13 are consequently greater than unity.

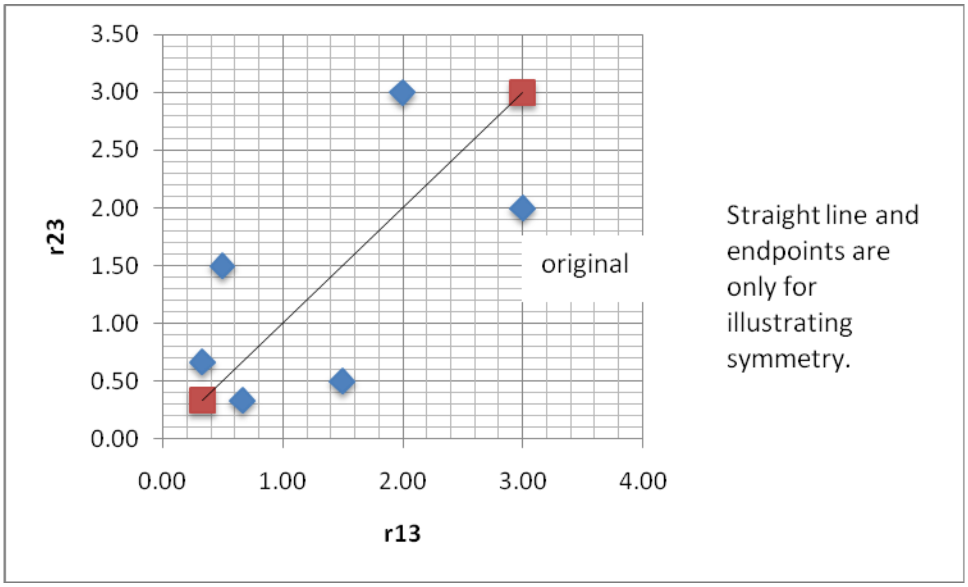

Figure 1. *Six allowed ordered pairs of ratios plotted for 3 objects. Unity slope line plotted as visual aid.*

So it can be seen that the new observations fall at any of six distinct positions on the graph (including the original for the case where the objects have not 'moved' in the pattern; i.e., they retain the same relative ranking of "distance metric" (e.g., root-sum-square of latitude and longitude).

The graph is repeated below with lines connecting the original observations ($R_{13} = 3$; $R_{23} = 2$) to the possible new ratios (with only one allowed for a single re-observation). These lines, numbered 2 through 6, have associated P3 permutations: Line 2 = (12), Line 3 = (312), Line 4 = (13), Line 5 = (231), Line 6 = (23). So an operator can plot the original and newly-observed ratios, connect the points with a line, and establish the exchange(s) of objects. We believe that an extension of the technique to four and more closely-space objects is possible.

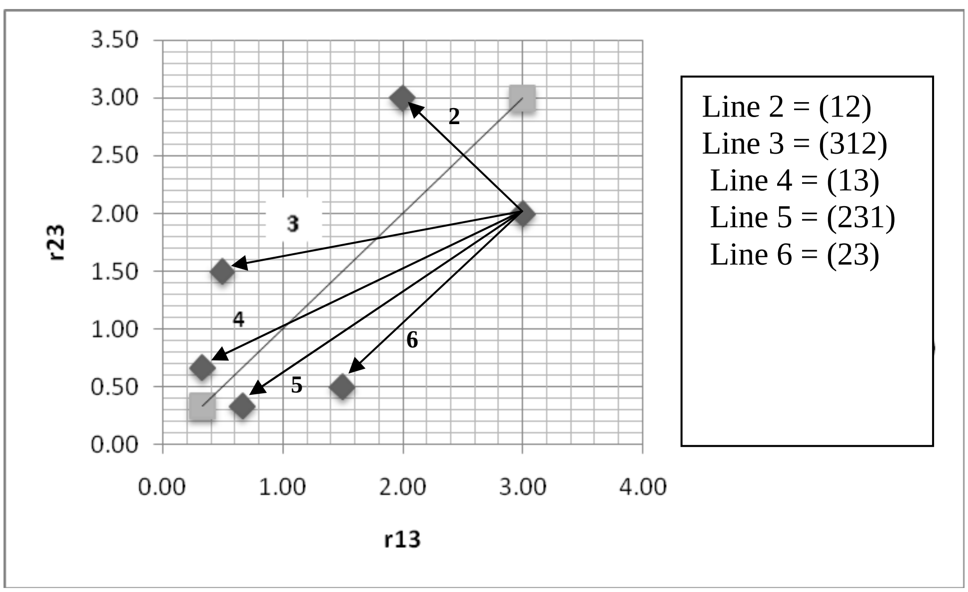

Figure 2. *Same as for Fig. 1, but illustrating one of several cases of before and after measurements. Here, the 'before' measurement is the right-most point, the 'after' can be any of the six points. The line codes show the corresponding permutation, indicating objects exchanged with respect to metric position ranking.*

This method can be verified with existing infrared spectrometer data. Below are shown three raw spectra (no instrumental or atmospheric corrections) obtained (using capabilities described in [1]) for blackbodies at 220, 225, & 230 C. It appears that the three have more in common than different, since they are dominated by the same pixel non-uniformities and the same atmospheric CO_2 absorption. So they are ideal candidates for demonstrating the raw signal ratioing technique (point by point for two spectra at a time), and generating the six ordered pairs of ratios having a common signal in the denominator. For this simple example, we take only 3 wavelength bins near 4.5 microns, and form a single product of the three ratios (this is an extension of a single band ratio that has the required property that $R_{ij} = 1/R_{ji}$). We have done the same for wavelengths in the CO_2 absorption feature to demonstrate the robustness of this ratio method to atmospheric attenuation.

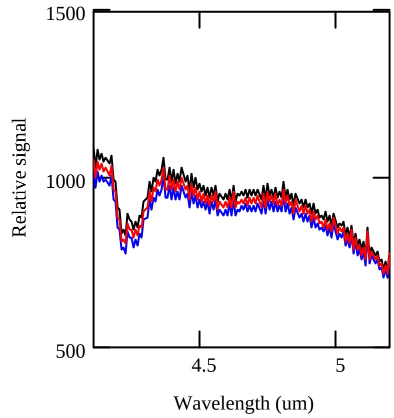

Figure 3. *Three uncorrected blackbody spectra at three temperatures: 220, 225, & 230 C.*

Plotting the six ordered pairs of signal ratio (again, these are the only ones possible for any permutation of the 3 objects giving rise to the three spectra), we see that the data points are well-separated on the plot, show the expected symmetry, and provide confidence that before and after measurements of this kind on three objects could be unambiguously identified on the basis of the permutation connecting the before and after points. (Note, since these are actual measurement of sources, i.e., blackbodies, with an infrared spectrometer, noise and instrumental defects are all included in the overall result. The center point in the figure represents three observations of the same blackbody temperature, giving a useful indication of the level of noise in the approach, here very small.)

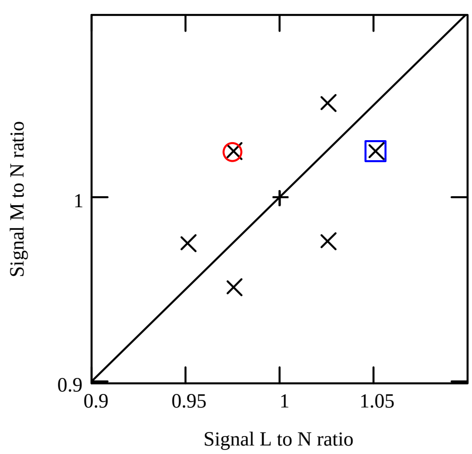

Figure 4. *Ordered pairs of actual signal ratios ($r_{L,N}, r_{M,N}$). Despite the small differences actual spectra (think: "similar brightness as observed with a metric sensor") shown in Figure 3, the ratio permutations are strongly differentiated. Blue box and red circle denote one case of transition from a "before" to an "after" configuration, respectively.*

3. THEORY & METHOD OF INTERPRETATION OF PLOTTED RATIOS

Here is how we propose to interpret two observations (denoted before and after "configurations", hereafter) on a ratio plot, as in Fig. 4. We relate each to a reference point on the figure, the upper-most data point, designated Po. Let l, m, and n represent the permutation such that the 'generalized signal' strengths are in increasing order ($l < m < n$). So Po is the ordered pair (m/l, n/l), since both ratios are greater than unity, and the slope (n/m) is also greater than unity. This is referred to as the identity permutation in what follows (even though we may never actually observe it in practice). Now let's examine the other two permutations about the unity slope line; these also have slopes greater than one. What is the permutation corresponding to the smaller of the two, on the lower left? It must be characterized by the ratios (l/n, m/n), the ordered pair having both ratio values less than one and slope (m/l) greater than one. The associated permutation is (312), one of the two cyclic permutations. What is the permutation corresponding to the other point? The ratio pairs must be (l/m, n/m), since the first ratio is less than one and the slope (n/l) is greater than one (and also the steepest of all possible). The corresponding permutation is (12), the exchange of the first and second objects (relative to Po). The sub-unity-slope points on the lower right are mirror images of the three points discussed above, and are obtained by exchanging the x and y ratios.

Based on the above observations and definition of the "identity" point Po, we suggest the following means to interpret before and after measurements as shown on Fig. 4. The six points are numbered starting with 0 for the identity and increasing clockwise to 5. Table 2 below presents the numbering and the corresponding permutation relative to the identity.

Table 2. Definition of relative permutations.

"Configuration"	Permutation (relative to 0)	Comments
0	Identity	Ratios > 1; slope > 1
1	(23)	Ratios > 1; slope < 1
2	(23)(12) [= 231]	Mixed ratios; slope < 1
3	(13)	Ratios < 1; slope < 1
4	(23)(13) [= 312]	Ratios < 1; slope > 1
5	(12)	Mixed ratios, slope > 1

In the table above, we have expressed Configurations 2 and 4 as the product of (23) and the mirror image points (Configurations 5 and 1, respectively) on the ratio plot. This emphasizes the exchange of the x and y ratio values for jumps between the mirror image points.

Now, given the nice group properties of permutations, we can determine their "transition permutation" that takes the 'before' observation to the 'after' observation (of two configurations) as follows. (1), determine the "relative permutation" (relative to Configuration 0) of the 'before' observation. (2) Do the same for the 'after' observation. (3) derive the permutation that takes 'before' to 'after'. Since $P_{ij}P_i = P_j$ is the equation for describing the transition from Configuration i (P_i) to Configuration j (P_j) by the permutation P_{ij}, we can write …

$$P_{ij} = P_j \, P_i^{-1},$$

… to solve for the transition permutation. (Each of the permutations in Table 2 is equal to its inverse, except for the two cyclic permutations, which have each other as inverses.) For example, say we observe first the objects in Configuration 5, and then in Configuration 3. Multiplying the permutations on the right side of Eqn. 1, (13)(12), from Table 2, one obtains 312. So there has been a cyclic permutation of the objects in going between these two configurations, that is, object 1 moved to Object 2's position, Object 2 to Object 3, and Object 3 took the place of Object 1.

Note that there are a total of 36 possible transitions between the 6 configurations, so the easy approach shown above helps to deal efficiently with the large number of possibilities. (These 36 possible transitions are each one of the six S3 permutations.)

4. Generalized signal ratios

In addition to broadband signals observed for objects in a configuration, other kinds of "signals" that satisfy the Rij = 1/Rji criterion include spectral slope endpoint ratios (where endpoint signals are derived from a curve fit to extract maximal information from the hyperspectral data) and even from a spectral curvature ratio. A curvature ratio of the form $(S1*SN)^{0.5}/Smid$, where the square root is the geometric mean between endpoint S1 and SN, and Smid is the actual observed (curve-fitted) midpoint, satisfies the $R_{ij} = 1/R_{ji}$ requirement. (Curiously, a straight line has curvature if defined this way, but that's of no consequence for our application.) A curvature plot using the three blackbodies observed with the spectrometer appears in Fig. 5, below.

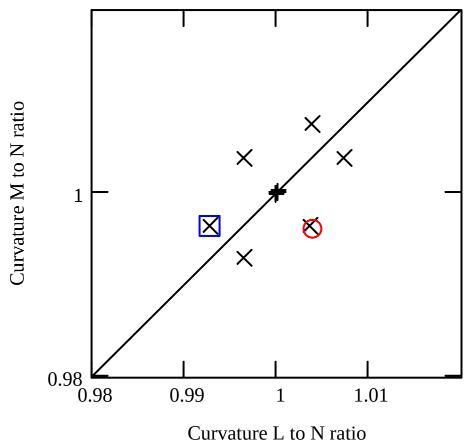

Figure 5. *Curvature ratios for six object configurations. This approach could be of use for objects signal levels that provide inadequate discrimination capability. The center point represents three observations of the same blackbody temperature, giving a useful indication of the level of noise in the approach. Blue square: starting configuration (temperatures 230, 220, 225); red circle: ending configuration (temperatures 225, 230, 220).*

Now we highlight the truly unique capability of the interpretation of "before and after" observations suggested here. The before and after points in Fig. 4 are at different positions that those of Fig. 5, although the before and after configurations are the same. (The relative curvature and signal levels are not positively correlated for blackbodies over this range of wavelengths.) The defining configuration of the uppermost point also differs in Figs. 4 and 5 – a different configuration provides the maximum ratio values (Po) in the two cases. The "transition permutation" can nevertheless be calculated (with $P_{ij} = P_j P_i^{-1}$) for both signal and curvature ratios. We find that these two radically different "generalized signals" give precisely the same change in configuration: the cyclic permutation (312).

5. CONCLUSIONS

We have shown a general method for keeping track of variable configurations of 3 objects, using a "generalized" signal ratio. The signal ratio can be a traditional broadband radiometric signal, but for objects of similar brightnesses that are confused by standard metric tracking sensors, would more likely be a multi- or hyper-spectral slope or curvature ratio. The latter variations of ratio type can be defined to ensure that signal ratios have the property that $R_{ji} = 1/R_{ij}$. These signal ratios are independent of atmospheric transmission since the objects share common lines of sight through the atmosphere, and are observed on timescales over which atmospheric transmission is relatively constant. By plotting ordered pairs of ratios (R_{ml}, R_{nl}) for configurations of Objects l, m, and n observed before and after a change, we can easily determine the permutation corresponding to the change. The method seems extendable to 4 or more confused objects. In the case of 4 objects, 24 configurations are possible, there are a total of 576 transitions among the 24 objects (24 of which involve no change at all), and a triplet of ratios (m/l, n/l, p/l) for before and after configurations that is plotted in a 3-dimensional space to determine the transition permutation.

6. REFERENCE

1. P.D. LeVan & D. Maestas, "A 3.5 to 12 Micron "Dualband" Spectrometer", *Optical Engineering* Vol 43, No. 12, page 3045 (2004).