DEMONSTRATION OF THE DSST STATE TRANSITION MATRIX TIME-UPDATE PROPERTIES USING THE LINUX GTDS PROGRAM

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ABSTRACT

The semi-analytical theory for the motion of a space object replaces the conventional equations of motion with two formulas: (1) equations of motion for the mean equinoctial elements, and (2) expressions for the short periodic motion in the equinoctial elements. Very complete force models have been developed for the mean element equations of motion and for the short periodic motion. There is also a semi-analytical theory for the partial derivatives of the perturbed motion. An interpolation strategy greatly assists in producing the mean elements, the osculating elements, the perturbed position and velocity, and the partial derivatives at the output request times. The semi-analytical satellite theory has been interfaced with a variety of batch least squares and Kalman Filter estimation processes. The current effort improves the software implementation of the semi-analytical theory for the partial derivatives so that (1) the mean element state transition matrix can be integrated backwards in time consistent with the interpolation architecture and (2) the epoch in a mean element orbit determination process can have an arbitrary location in a span of observation data. Both of these new capabilities support studies of the propagation of the state error covariance in the mean equinoctial elements. The paper describes the mathematical formulation and the software implementation in the Linux GTDS DSST program, and provides several test cases to illustrate the capabilities.

1. INTRODUCTION

The semi-analytical theory for the motion of a space object replaces the conventional equations of motion with two formulas [1]:

1. Equations of motion for the mean elements

2. Expressions for the short periodic motion

The intent of the semi-analytical theory is that the very small integration grid of the Cowell numerical integration (on the order of hundreds of steps per orbital revolution) be replaced with a much larger step (on the order of one or two steps per day). Such large steps are very computationally efficient. Also, the motion of the non-singular equinoctial mean elements is more linear and this has positive implications for orbit determination processes based on the semi-analytical theory. The semi-analytical theory includes a comprehensive interpolation strategy.

Andrew Green [2] developed a general semi-analytical theory for the partial derivatives of the perturbed motion that is compatible with the semi-analytical theory for the motion. The primary emphasis in Green's work was on weighted least squares orbit determination processes. The perturbed partial derivatives are expressed by

$$G = \left[\left\{ \frac{\partial \underline{a}^*(t)}{\partial \overline{\underline{a}}_0} \right\} \quad \left\{ \frac{\partial \underline{a}^*(t)}{\partial \underline{c}} \right\} \right] \tag{1}$$

where G is a 6 x l matrix (l is the number of solve-for parameters). The vector $\underline{a}^*(t)$ is composed of the osculating equinoctial elements at an arbitrary time, t. Green assumed that the epoch time was at the beginning of the observation time interval. The vector $\overline{\underline{a}}_0$ is composed of the non-singular equinoctial mean elements at the epoch time. The vector is $\underline{c}$ is composed of the dynamical parameters (such as an atmospheric drag parameter or a solar radiation pressure coefficient). The G matrix can be expanded as

$$G = [I + B_1][B_2 \quad B_3] + [0 \quad B_4] \tag{2}$$

where

$$B_1 = \left[\frac{\partial \, \varepsilon \underline{\eta}_1(\overline{\underline{a}}^*)}{\partial \overline{\underline{a}}^*(t)} \right]_{6x6} \tag{3}$$

$$B_2 = \left[\frac{\partial \overline{\underline{a}}^*(t)}{\partial \overline{\underline{a}}_0^*} \right]_{6x6} \tag{4}$$

$$B_3 = \left[\frac{\partial \overline{\underline{a}}^*(t)}{\partial \underline{c}} \right]_{6x(l-6)} \tag{5}$$

$$B_4 = \left[\frac{\partial \, \varepsilon \underline{\eta}_1(\overline{\underline{a}}^*)}{\partial \underline{c}} \right]_{6x(l-6)} \tag{6}$$

The matrices B_1 and B_4 represent the short periodic portion of the semi-analytical partial derivatives. The B_2 and B_3 matrices are the partial derivatives of the mean elements at arbitrary time with respect to the solve-for parameters. The B_2 and B_3 matrices are governed by the linear differential equations:

$$\frac{d}{dt} B_2 = AB_2 \text{ with } [B_2]_{t_0} = I \tag{7}$$

$$\frac{d}{dt} B_3 = AB_3 + D \quad \text{with } [B_3]_{t_0} = [0]_{6x6} \tag{8}$$

where

$$A = \left\{ \frac{\partial\left[\frac{d}{dt}\left(\bar{a}^*\right) \right]}{\partial\underline{\bar{a}}^*(t)} \right\}_{6x6} \tag{9}$$

$$D = \left\{ \frac{\partial\left[\frac{d}{dt}\left(\bar{a}^*\right) \right]}{\partial\underline{c}} \right\}_{6x(l-6)} \tag{10}$$

where it is understood that $\frac{d}{dt}\left(\bar{a}^*\right)$ stands for the right hand side of the equations of motion for the mean elements.

The B_1 and B_4 matrices can be obtained by direct application of Eqs. (3) and (6). The form of the short periodic expansions is given in [1]. There is a comprehensive interpolation strategy for the partial derivatives that is analogous to the interpolation strategy for the satellite theory. The partial derivative capabilities developed by Green were implemented in the GTDS R&D orbit determination program and tested via double-sided finite differencing techniques. Generally the short periodic contributions to the partial derivatives are small.

Subsequently Taylor [3], Wagner [4], and Herklotz [5] developed recursive filters to directly estimate the mean elements from the observation data. Taylor and Wagner developed and tested an Extended Semi-analytical Kalman Filter (ESKF) that reconciles the conflicting goals of the perturbation theory (very large stepsizes) and the Extended Kalman Filter (EKF) (re-linearization at each new observation time). Herklotz employed the Square Root Information Filter (SRIF) due to Bierman [6] in order to have the flexibility to process large numbers of observations. Semi-analytical SRIF filter algorithms which solve for the precision mean elements were developed. Herklotz tested his algorithm with simulated crosslink ranging data for an eight satellite constellation with four equatorial 24-hr orbits and four inclined 24-hr orbits. The ESKF and the Semi-analytical SRIF algorithms taken together successfully employ various forms of the perturbed partial derivatives.

In 2008, Folcik [7] developed a Backward Smoothing Extended Kalman Filter (BSEKF) for orbit determination which employed the Semi-analytical satellite theory. This filter updates several previous time values of the mean element state at each step and introduces additional requirements for the mean element state transition matrix. The major new requirement was to integrate the state transition matrix backward in time.

More recently [8], there has been strong interest in the propagation of state error covariance in the mean equinoctial element coordinates. Covariance can be propagated using the state transition matrix. Specifically, the requirements that motivated the current work are the following:

- Modification of the GTDS R&D orbit determination program so that the mean element state transition matrix B_2 can be integrated backwards in time using Eq.(7) and consistent with the interpolation architecture previously developed for the Semi-analytical Satellite Theory.

- Modification of the GTDS R&D orbit determination program so that the epoch in a Semi-analytical Satellite Theory Differential Correction step can occur later in time than some or all of the observation data

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The roadmap of this paper is as follows. In Section 2, we describe the GTDS DSST source code modifications needed to allow the backwards integration functionality. We also describe the modifications required to allow backwards integration or a combination of forward and backwards integration while running the differential correction (DC) subprogram with DSST. In Section 3, we describe the DSST orbit propagation test cases required to exercise the new capability. These include orbit propagation with state transition matrix propagation. The results of forward and backward propagation of the state transition matrix are combined in a test checking the semigroup property of the state transition matrix. In Section 4 we describe the DSST orbit determination test cases. The test cases demonstrate the location of the solve-for state epoch at various places in the observation span. Conclusions and Future Work end the paper.

2.   GTDS DSST SOURCE CODE MODIFICATIONS

Linux R&D GTDS is a comprehensive, multi-functioned orbit determination system that is maintained under configuration control by the authors. Linux R&D GTDS currently employs the Fortran 77 programming language. Linux R&D GTDS originally stems from the efforts at the Draper Laboratory and by graduate students of the MIT Aeronautics and Astronautics Department from 1979 onward. References [7] and [9] through [14] describe the evolution of GTDS R&D in the MIT community. This was aided by the efforts of AFRL personnel from the mid 90's onward. From 2001 onward, MIT Lincoln Laboratory personnel have been involved in the maintenance of R&D GTDS. More recently, Pacific Defense Solution (PDS) personnel supporting AFRL have participated.

The source code modifications needed to allow the GTDS/DSST backwards integration functionality involved several subroutines. These subroutines were RESWRV, SNGSTP, ORBITV, and SKFPRT. The top level GTDS ephemeris generation driver ORBIT calls RESWRV to reinitialize the integration after a change in direction. Subroutine SNGSTP initiates and executes the Runge-Kutta integrator and also calls the short periodic coefficient generation process (SPGENR) for the GTDS DSST. Subroutine ORBITV provides the output at request time functionality for the DSST. Subroutine SKFPRT is concerned with the computation of partial derivative matrices via short arc interpolation and the averaged interpolator. This improves the efficiency in runs with high data rate sensors.

In order for backwards Semi-analytical integration to function for ephemeris generation, subroutines ORBITV and SKFPRT required modifications. There were several conditional statements that were used to detect whether enough time had progressed in the integrator in order to recalculate interpolation coefficients. These conditional statements were modified to correctly detect the passage of time while time was progressing backward as well as forward.

In order to allow backward integration or a combination of forward and backward integration while running the differential correction (DC) subprogram, substantial modifications were made to the RESWRV subroutine. This subroutine was called by the ORBIT subroutine when integration had to restart because a change in the direction of integration was needed. This situation occurred, for example, when processing observations in the DC subprogram. If the set of observations included observations both before and after the epoch time of the initial orbital conditions, the DC subprogram would first propagate backwards to process the observations that were before the epoch time and would then change direction to propagate forward to process the observations that were after the epoch time. Because the RESWRV subroutine did not include code to handle a change in integration direction for the DSST propagator, several statements were added to perform the necessary tasks. These tasks included: (1) changing the sign of the integration stepsize variable, (2) resetting arrays used for quadrature and orbit element partial derivatives, (3) re-computing the Greenwich Hour Angle, and (4) calling SNGSTP to restart the integrator in the new direction. In subroutine RESWRV, the SNGSTP call was made with a particular value of the IERR argument. SNGSTP was

modified to check for that value of IERR so that an unnecessary call to AVRINT that would usually be made in SNGSTP was avoided. The SKFPRT and ORBITV modifications ensure that the state transition matrix is correct when doing backward propagation. Some smaller scale changes were required for the Differential Correction program to correctly process space-based observations.

3. DSST ORBIT PROPAGATION TEST CASES

The following approaches are employed in testing the new backwards in time integration and partial derivative capabilities:

4. Closure test of the backwards/forward integration of the mean element equations of motion (Test 1 and 1B)

5. Closure test of the backwards/forward integration of the full semi-analytical theory (mean element equations of motion plus short periodic model) (Test 2 and Test 2B)

6. Comparison of the mean element state transition matrices (B_2) computed with backwards and forward integration via the semigroup property and its corollary (Test 3 and Test 3B)

Closure Test with Mean Elements Only (Test Case 1 and 1B)

This test starts with a 10-day forward propagation of the mean elements. The epoch and epoch mean elements for this propagation are given in Table 1.

Table 1. Epoch Mean Elements for the Forward in time Orbit Propagation (Test Case 1)

Orbit Element	
Semi-major axis	6706.9662 km
Eccentricity	0.0010252154D0
Inclination	87.266393 deg
Longitude Ascending Node	64.668178 deg
Argument of Perigee	94.431363 deg
Mean Anomaly	105.69973 deg
Epoch (UTC)	2008 Sept 15, 21 h 59 m 46 s

The coordinate system usage in this test case is as follows:

1. The epoch mean element set is in J2000 coordinates

2. The integration of the mean elements is carried out in J2000 coordinates

3. The output mean element sets are in J2000 coordinates

This choice of coordinate systems is designed to avoid the proliferation of coordinate systems which would occur if the epoch mean elements were assumed to be in TOR coordinates.

The mean element integration is carried out using the Runge-Kutta integration process over an interval of 10 days. The mean element element interpolator interval is 1 day and the 3-pt Hermite interpolator algorithm is employed.

The mean element dynamics includes the J2 and J2-squared terms.

The output mean elements at the 10-day time are given in Table 2.

Table 2. Ten Day Mean Elements for the forward in time propagation (Test Case 1)

Orbit Element	
Semi-major axis	6706.966200 km
Eccentricity	0.1025391325E-02
Inclination	87.26465414 deg
Longitude Ascending Node	60.62245164 deg
Argument of Perigee	53.13841140deg
Mean Anomaly	84.86686503 deg
Output (UTC)	2008 Sept 25, 21 h 59 m 46 s

The output elements exhibit the expected secular motion in the Longitude Ascending Node, Argument of Perigee, and Mean Anomaly.

The elements in Table 2 are then input into the GTDS DSST orbit propagator and the mean element integration is run backward in time for ten days. This is Test 1B. The input file for this backward in time integration is given in Fig. 1:

```
CONTROL   DATAMGT
OGOPT
POTFIELD  1 11
END
FIN
CONTROL   EPHEM
EPOCH              1080925.0              215946.0
ELEMENT1 11  6  1  6706.9662              0.001025391325         87.26465414
ELEMENT2           60.62245164           53.13841140            84.86686503
OUTPUT   11  2  1  1080915.0              215946.0               43200.0
ORBTYPE   5  1 11  43200.0                1.0
OGOPT
GMCON     1        398600.436D0
BODYRAD   1        6378.137
CNM       3  2  0  -0.0010826256063587
MAXDEGEQ  1        2.0
MAXORDEQ  1        0.0
SPOUTPUT       1  1.0
NCBODY    1
SCPARAM           3.1415D-6              685.D0
SPSHPER   1
AVRHARM                                                         1.0
END
FIN
```

Figure 1. GTDS Input file for the backward in time integration process

The 10 day output for the backward in time integration process is given in Table 3.

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Table 3. Ten Day Mean Elements for the Backward in Time Propagation (Test Case 1B)

Orbit Element	
Semi-major axis	6706.966200 km
Eccentricity	0.1025215399E-02
Inclination	87.26639300 deg
Longitude Ascending Node	64.66817800 deg
Argument of Perigee	94.43136300deg
Mean Anomaly	105.6997300 deg
Output (UTC)	2008 Sept 15, 21 h 59 m 46 s

Comparison of Tables 1 and 3 shows tight closure between the forward and backward mean element integration processes.

Closure Test with Mean Elements and Short-Periodics (Test Case 2 and 2B)

This test case repeats the previous test case **with the short-periodic model enabled**. The forward integration (Test Case 2) again employs the epoch mean element set given in Table 1. The process is the same as Test Case 1 except that the short-periodic model and the short-periodic Fourier coefficient interpolator construction process are exercised on the mean element integration grid. The general, recursive first order zonal short-periodic model due to Slutsky [15] is used to generate the J_2 short-periodic coefficients. The Lagrangian process is used to generate the short-periodic coefficient interpolators. The resulting mean element and short-periodic coefficient interpolators are exercised at each output time (once per hour over the 10 days). Again, the J2000 coordinate system is used throughout the test case.

Table 4. Epoch Perturbed Position and Velocity for the Forward Integration (Test Case 2)

Coordinate	
x-position	-2595.256643 km
y-position	-5741.664984 km
z-position	-2321.359682 km
x-velocity	1.450193597 km/sec
y-velocity	2.258205121 km/sec
z-velocity	-7.221683085 km/sec
Output (UTC)	2008 Sept 15, 21 h 59 m 46 s

The backward integration (Test 2B) uses the mean element set given in Table 2 as the initial values.

The GTDS input file for this backward in time integration is given in Figure 2:

```
CONTROL   DATAMGT
OGOPT
POTFIELD  1 11
END
```

```
FIN
CONTROL   EPHEM
EPOCH              1080925.0           215946.0
ELEMENT1 11  6  1  6706.9662           0.001025391325      87.26465414
ELEMENT2           60.62245164         53.13841140         84.86686503
OUTPUT   11  2  1  1080915.0           215946.0            3600.0
ORBTYPE   5  1 11  43200.0             1.0
OGOPT
GMCON     1        398600.436D0
BODYRAD   1        6378.137
CNM       3  2  0  -0.0010826256063587
MAXDEGEQ  1        2.0
MAXORDEQ  1        0.0
SPOUTPUT     1  1.0
NCBODY    1
SCPARAM           3.1415D-6           685.D0
SPSHPER   2
AVRHARM                                                   1.0
END
FIN
```

Figure 2. GTDS Input file for the backward in time integration process (Test 2B)

The 10 day output for the backward in time integration process is given in Table 4.

Table 5. Output Perturbed Position and Velocity for the Backward Integration (Test Case 2B)

Coordinate	
x-position	−2595.256644 km
y-position	−5741.664983 km
z-position	−2321.359682 km
x-velocity	1.450193597 km/sec
y-velocity	2.258205121 km/sec
z-velocity	−7.221683085 km/sec
Output (UTC)	2008 Sept 15, 21 h 59 m 46 s

We observe that the closure between the forward and backward DSST integration processes with the short-periodic model on is on the order of 1 mm.

We also reviewed the output at time points off the interpolator construction time grid and observed a similar level of closure.

Testing of the Mean Element State Transition Matrices Computed with the Forward and Backward Integration Processes (Test Case 3 and 3B)

This test case repeats Test Case 2/2B **with the mean element state transition matrix functionality enabled**. The forward integration (Test Case 3) again employs the epoch mean element set given in Table 1. The mean element state transition matrix is integrated using Equation (7). The Runge-Kutta integration process is used and the original mean element equation of motion integration grid is also used for the state transition matrix ordinary differential equations. Since the state transition matrix rates are available, we can employ a Hermite interpolation process to construct the state transition matrix interpolators. The resulting mean element, short-periodic coefficient, and state

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transition matrix interpolators are exercised at each output time (once per hour over the 10 days). Again, the J2000 coordinate system is used throughout the test case.

The backward integration (Test Case 3B) uses the mean element set given in Table 2 as the initial values.

In both Test Cases 3 and 3B, the mean element state transition matrix is initialized with the Identity matrix (see Eq. 7).

The input file for the forward in time integration with state transition matrix is given in Figure 3:

```
CONTROL    DATAMGT
OGOPT
POTFIELD  1 11
END
FIN
CONTROL    EPHEM
EPOCH              1080915.0            215946.0
ELEMENT1 11  6  1  6706.9662           0.0010252154D0       87.266393
ELEMENT2           64.668178           94.431363            105.69973
OUTPUT   11  2  1  1080925.0           215946.0             3600.0
ORBTYPE   5  1 11  43200.0             1.0
OGOPT
GMCON      1       398600.436D0
BODYRAD    1       6378.137
CNM        3  2  0  -0.0010826256063587
MAXDEGEQ  1        2.0
MAXORDEQ  1        0.0
SPOUTPUT         1  1.0
NCBODY     1
SCPARAM            3.1415D-6            685.D0
SPSHPER    2
AVRHARM                                                     1.0
SSTESTFL  1
SSTAPGFL  1
STATEPAR  3
STATETAB  1  2  3  4.0                 5.0                  6.0
SSTESTOU     1
END
FIN
```

Figure 3. GTDS Input file for the forward in time integration process with state transition matrix (Test 3)

While the state transition matrices from Test Cases 3 and 3B don't close in the same way that the trajectories do, they are connected by the semi-group property [3].

Let the mean element state transition matrix from the forward integration process be denoted by $\Phi(t,t_0)$, where t is an arbitrary output time and t_0 is the initial epoch. Let the mean element state transition matrix from the backward integration process be denoted by $\Psi(t,t_{final})$, where t is an arbitrary output time and t_{final} is the final epoch (in our case 10 days after the initial epoch).

The semi-group property dictates that

$$\Psi(t_0,t_{final}) = [\Phi(t_{final},t_0)]^{-1} \tag{11}$$

The mean element state transition matrix (forward integration) at t=10 days was entered into Matlab as:

```
A(1,1) = [1.000000000000e+00]
A(1,2) = [0.0e+00]
A(1,3) = [0.0e+00]
A(1,4) = [0.0e+00]
A(1,5) = [0.0e+00]
A(1,6) = [0.0e+00]

A(2,1) = [-1.705570484467e-07 ]
A(2,2) = [ 7.034768292492e-01 ]
A(2,3) = [-7.107199186497e-01 ]
A(2,4) = [-4.341053167104e-04 ]
A(2,5) = [-2.057714946580e-04 ]
A(2,6) = [ 0.000000000000e+00 ]

A(3,1) = [-3.870317162654e-07 ]
A(3,2) = [ 7.107197529401e-01 ]
A(3,3) = [ 7.034735095352e-01 ]
A(3,4) = [-9.850808568351e-04 ]
A(3,5) = [-4.669412009957e-04 ]
A(3,6) = [ 0.000000000000e+00 ]

A(4,1) = [ 1.696212698224e-05 ]
A(4,2) = [-4.760707514880e-05 ]
A(4,3) = [ 1.245109740282e-04 ]
A(4,4) = [ 1.641878389035e+00 ]
A(4,5) = [ 2.358821484957e-01 ]
A(4,6) = [ 0.000000000000e+00 ]

A(5,1) = [-3.016668060990D-05 ]
A(5,2) = [ 8.466784713193D-05 ]
A(5,3) = [-2.214393800358D-04 ]
A(5,4) = [-1.076343176998D+00 ]
A(5,5) = [ 4.544245565769D-01 ]
A(5,6) = [ 0.000000000000D+00 ]

A(6,1) = [-2.213146568294D-01 ]
A(6,2) = [-1.953402807553D-03 ]
A(6,3) = [ 5.108909527527D-03 ]
A(6,4) = [ 8.526135429444D-01 ]
A(6,5) = [ 4.041498979315D-01 ]
A(6,6) = [ 1.000000000000D+00 ]
```

We can then use the Matlab matrix inversion command **inv(A)** to estimate the mean element state transition matrix from the backwards integration process. The command **inv(A)** gives:

Columns 1 through 3

```
 1.000000000000000e+00                          0                          0
 3.950535127678441e-07      7.034735102802463e-01      7.107199194023942e-01
 1.510495832951891e-07     -7.107197536927949e-01      7.034768299942281e-01
-1.482378846775489e-05      1.066049100030003e-04     -4.697861038582019e-05
 3.127295131763229e-05     -2.248986333844597e-04      9.910829027270199e-05
 2.213146568321223e-01      5.005170020777697e-03     -2.205677199161744e-03
```

Columns 4 through 6

```
                    0                         0                         0
 9.699288967338311e-04     5.453717075001824e-04                       0
 3.708544499813338e-04     2.085241742718125e-04                       0
 4.544245565771885e-01    -2.358821484958433e-01                       0
 1.076343176998657e+00     1.641878389036014e+00                       0
-8.224525163074912e-01    -4.624486689989787e-01    1.000000000000000e+00
```

Comparison of **inv(A)** with the mean element state transition matrix computed in the backwards integration process shows 8 to 9 digits of agreement. This accuracy is consistent with the manual entry the forward integration state transition matrix from GTDS into Matlab.

4. DSST ORBIT DETERMINATION TEST CASES

The following orbit determination cases demonstrate the application of the new GTDS DSST capability (backward in time integration) to support arbitrary location of the solve-for vector epoch in an observation span.

For each distinct satellite case, we developed the following tests:

1. DSST Differential Correction with the epoch at the beginning of the observation span

2. DSST Differential Correction with the epoch at the end of the observation span

3. DSST Differential Correction with the epoch at an intermediate (usually near the middle) point in the observation span

Two satellite cases are employed:

1. GPS case with observations from November 2008; the observations are actual position coordinates generated by the National Geospatial-Intelligence Agency [NGIA] [16, 17]

2. Experimental Geodetic Payload (EGP) case with observations from August and September 2002; the observation are actual SLR ranges from approximately 20 ground stations [ILRS][18]. The EGP is in a near circular orbit at 1488 km altitude with a 50 degree inclination. The EGP spacecraft is a hollow sphere covered by mirrors and corner reflectors.

GPS Orbit Determination Test Cases

All of the GPS test cases employ the same observation data from November 2008. The observations are Earth-Centered Earth-Fixed (ECEF) position coordinates. The ECEF coordinates are a standard observation input format for GTDS [10]. The a priori quality of this data was assumed to be 10 meters.

All of these cases employ the same physical models and DSST truncations:

Dynamical Models:

• 16 x 16 geopotential – GRACE Gravity Model (GGM02C)

• lunar-solar point masses

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- solar radiation pressure (spherical s/c model)

DSST Truncations

- Averaged Perturbation Models

    16 x 0 zonal harmonics

    tesseral resonance due to the even order harmonics

    J2-squared terms

    lunar-solar point masses

    solar radiation pressure (time-independent numerical model)

- Short-periodic model

    zonal harmonic terms

    tesseral m-daily terms

    tesseral linear combination terms

    lunar-solar point masses

    J2-squared terms

    J2/tesseral m-daily coupling

    solar radiation pressure

The solve-for vector includes the mean equinoctial elements and the solar radiation parameter. The state transition matrix dynamics include the J2 terms. The solar radiation pressure partial derivatives are obtained by numerical integration of Eq.(8). The mean element equation of motion integration grid is used for the integration of Eq.(8).

The details of the GTDS DSST input data file for the GPS DC are illustrated in Appendix A (for Test Case 9). The DSST User Guides [19, 20] are useful in understanding this file.

Table 6. GTDS DSST Orbit Determination Test Cases for the GPS satellite (Test Cases 9, 10, and 11)

Parameter	Test Case 9	Test Case 10	Test Case 11
Epoch of the solve-for vector within the observation span	Epoch at the beginning of the observation span	Epoch at the end of the observation span	Epoch near the middle of the observation span
Iterations to DC convergence	8	9	4

Converged semi-major axis standard deviation	1.3 cm	1.3 cm	1.33 cm
Converged solar radiation pressure coefficient	0.2112312D+01	0.2112381D+01	0.2100249D+01
Converged solar radiation pressure coefficient standard deviation	0.728D-02	0.728D-02	0.750D-02
Converged position error RMS, meters	3.1970863	3.2141006	3.3499348
Initial Weighted RMS	143957.34	401397.17	14.230453
Converged Weighted RMS	0.18724843	0.18729642	0.19369487
Number of observations available	579	579	579
Number of observations accepted	571	571	574
Mean residual x, meters	-0.2576	-0.2552	-0.2339
Mean residual y, meters	8.3581E-02	8.4007E-02	0.2729
Mean residual z, meters	-8.1697E-02	-8.1997E-02	-9.0637E-02

Our goal with these test cases was to demonstrate that the epoch in a semianalytical DC can be located anywhere in an observation interval without significantly impacting the quality of the fit. There are small variations; we think some of these relate to different errors in the apriori vector errors. We note that the number of available observations is the same for all three cases. We note that the percentage of accepted observations is very high (98 or 99%) for all three cases. This expected due to the quality of the GPS ephemerides that are used as data.

We also note that two days of GPS precise ephemeris can be accurately approximated by a single mean equinoctial element set.

EGP Orbit Determination Test Cases

All of the EGP test cases employ the same observation data from August and September 2002. The observations are Satellite Laser Ranging (SLR) ranges. The range data are a standard observation input format for GTDS. The apriori quality of this data was assumed to be 2 meters. The observation span is 10-days in length.

All of these cases employ the same physical models and DSST truncations:

Dynamical Models:

- 30 x 30 geopotential – EGM96 Gravity Model

- lunar-solar point masses

- solar radiation pressure (spherical s/c model)

- atmosphere drag (Jacchia-Roberts)

- solid Earth tides

DSST Truncations

- Averaged Perturbation Models

    30 x 0 zonal harmonics

    tesseral resonance due to tesseral coefficient pairs (25,25) through (30,25)

    J2-squared terms

    lunar-solar point masses

    solid Earth tides

    atmosphere drag (time independent numerical model)

    solar radiation pressure (time-independent numerical model)

- Short-periodic model

    zonal harmonic terms

    tesseral m-daily terms

    tesseral linear combination terms

    lunar-solar point masses

    J2-squared terms

    J2/tesseral m-daily coupling

    solar radiation pressure

The solve-for vector includes the mean equinoctial elements and the solar radiation parameter. The state transition matrix dynamics include the J2 terms. The solar radiation pressure and atmosphere drag partial derivatives are

obtained by numerical integration of Eq.(8). The mean element equation of motion integration grid is used for the integration of Eq.(8).

The details of the GTDS DSST input data file for the EGP DC are illustrated in Appendix B (for Test Case 13). Among other things, these cases test GTDS operation with a large number of ground based sensors.

Table 7. GTDS DSST Orbit Determination Test Cases for the EGP satellite (Test Cases 13, 14, and 15)

Parameter	Test Case 13	Test Case 14	Test Case 15
Epoch of the solve-for vector within the observation span	Epoch at the beginning of the observation span	Epoch at the end of the observation span	Epoch at an intermediate point in the observation span
Iterations to DC convergence	7	5	5
Converged semi-major axis standard deviation	0.693 cm	0.432 cm	0.216 cm
Converged solar radiation pressure coefficient	0.14164574D+01	0.141610744D+01	0.142206157D+01
Converged solar radiation pressure coefficient standard deviation	0.179D-01	0.179D-01	0.180D-01
Converged atmosphere drag parameter	0.4593108D+01	0.459162142D+01	0.4590957405D+01
Converged atmosphere drag parameter standard deviation	0.613D-01	0.613D-01	0.613D-01
Converged position error RMS, meters	7.7115502	7.7117070	7.7122563
Initial Weighted RMS	13563.2	14.2	177.2
Converged Weighted RMS	3.9244741	3.9246078	3.926902
Number of observations	3580	3580	3580

available			
Number of observations accepted	3562 (99%)	3562 (99%)	3562 (99%)

These EGP test cases, like the previous GPS orbit determination test cases, demonstrate that the epoch in the semianalytical DC can be located at multiple locations within the observation interval without significantly impacting the quality of the fit. There are variations in the number of iterations required for convergence; we think that these relate to the different errors in the a priori vectors. We note that the number of available observations is the same for all three cases. We note that the percentage of accepted observations is very high (99%) for all three cases. This is expected due to the quality of the SLR ranges that are used as data.

We note that 10 days of EGP precise ephemeris can be accurately approximated by a single mean equinoctial element set.

5. CONCLUSIONS

The GTDS Semi-analytical Satellite Theory (DSST) architecture has been extended to allow backwards in time numerical integration. This capability has been interfaced with the mean element interpolator and the short-periodic coefficient interpolator. Computation and interpolation of the mean element state transition matrix has also been included. The overall interpolator functionality including initialization of the interpolators and advancement of the interpolators in time (both forward and backwards) has been tested.

The overall purpose of the GTDS DSST backwards in time capability is to increase the opportunities for comparison of GTDS DSST results with results obtained with other orbit determination programs. Specifically, the goal of the current effort was to demonstrate the backwards in time capability with the GTDS DSST Ephemeris Generation (EPHEM) and Differential Correction (DC) programs.

Several tests of the new capabilities have been completed. We start in Case 1 with an assumed mean element set at a given epoch and integrate this mean element set forward in time over a 10-day interval. We then record the mean elements at the 10-day point and use them as the epoch conditions in a backward integration (Case 1B). Our metric for this test is the closure of the backwards integration with the assumed epoch mean element set in Case 1. Almost exact closure is achieved.

Test cases 2 and 2B repeat the mean element integrations of Tests 1 and 1B except that the DSST short-periodic model is now engaged. Closure between the Test 1 and 1B perturbed state vectors at the initial epoch time are at the level of 1 mm. Tests 1/1B and 2/2B employ the J_2 and J_2-squared terms in the mean element equations of motion and the J_2 terms in the short-periodic model. So the full recursive models (PZONAL and SPZONL, respectively) are employed but the evaluation is truncated at J_2. The purpose of this modeling choice was to reduce the complexity of these tests.

Tests 3 and 3B are the same as tests 2 and 2B except that the mean element state transition matrix capability has been enabled. The perturbed trajectories are unchanged. While the state transition matrices from Tests 3 and 3B don't close in the same sense that the perturbed trajectories do, they are connected by the semi-group property [Eq.(11)]. The semi-group property was demonstrated in these tests at the level of eight (8) to nine (9) digits which

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was consistent the remainder of this process. Matlab was employed to achieve the matrix inversion required to demonstrate the semi-group property.

The two differential correction cases (Test Cases 9, 10 and 11 for the GPS case and Test Cases 13, 14, and 15 for the EGP case) provide further test of the backwards integration with time capability. Cases 9 and 13 are conventional DSST DC runs with the solve-for epoch at the beginning of the observation span (2 days for the GPS case and 10 days for the EGP case). Cases 10 and 14 put the solve-for epoch at the end of the observation span. Thus the DSST backwards integration capability is tested in the DC context with rather complete models for both the mean element equations of motion and the short-periodic model. We note that these cases (10 and 14) also test the interaction between the DSST backwards integration capability and several GTDS physical model binary files. Finally, Test Cases 11 and 15 place the solve-for epoch at an intermediate point in the observation span requiring the integration to go both backwards and forwards within a single DC iteration. It is satisfying that all three GPS cases and EGP cases give similar results, respectively.

Another way to look at these tests is with respect to how they exercise the partial derivative capability:

- Test Cases 3 and 3B just compute the B_2 matrix (see Eq. 7)

- Test Cases 9, 10, and 11 compute the B_2 matrix and the B_3 matrix (see Eq. 8). For the B_3 matrix, only the solar radiation pressure parameter partial derivatives are computed.

- Test Cases 13, 14, and 15 also compute the B_2 and B_3 matrices. Now the B_3 matrix includes both the solar radiation pressure and atmospheric drag parameter partial derivatives.

Both the GPS and EGP cases demonstrate that long arcs of observation data can be accurately compressed into a single nonsingular mean element set. Sequences of nonsingular mean element sets are of interest for studies that try to improve the prediction of the long term motion of such space objects.

Version control for the Linux GTDS R&D orbit computation program used in this study is maintained using the **subversion** utility [21]. Finally we have given several GTDS DSST input files as examples.

6. FUTURE WORK

None of the test cases completed for this paper actually exercise the finite differencing method. The A matrix in Eq. (9) is computed using just the closed-form J_2 terms (see GTDS DSST subroutine J2PART). For the solar radiation pressure and atmospheric drag parameter partial derivatives, the D matrix in Eq. (10) is just the portion of the mean element rates due to respective perturbation divided by the parameter. See Eq. (2-94) in Green [2]. But several finite differencing options are connected to the GTDS DSST keyword SSTAPGFL. These options should be tested using the methods and cases developed in this paper. If the finite differencing approach causes observable error, consideration should be given to analytical and quadrature approaches for reducing the dependence on finite differences.

We would like to develop a test for the B_3 matrices (parameter partials) that connects the values from the forward integration with the values from the backward integration. This would be analogous to the semi-group property test for the mean element state transition matrix.

The association of the DSST with the Runge-Kutta 4 integration stems from the initial build of the DSST in the early 1980s. At that time, time intervals of just a few days were the primary interest. We would like to investigate the application of Explicit Runge-Kutta Methods of Higher Order methods such as the Dorman & Prince 8(6) that is described in [22] to the DSST orbit propagator.

We would like to undertake covariance propagation tests. One case would use the DSST DC to fit a set of observations. The covariance would be recorded at the epoch. One can then use the DSST ephemeris program to propagate that covariance forward to some time in the future and then backward to the original epoch again. The covariance should be the same modulus any differences from numerical artifacts. The existing Extended Semianalytical Kalman Filter (ESKF) [3, 4] may also play a role in these tests.

Another test that could be run with the DSST DC is a demonstration that when one moves the initial state epoch from the beginning of the fitspan to the middle and then to the end of the fitspan that the covariance should be minimal at the center of the fitspan. This should match the covariance when propagated with EPHEM from the beginning to the end of the fitspan and from the end of the fitspan back to the beginning.

Finally, to demonstrate the value of the enhanced GTDS DSST functionality, we propose to develop a Spherical Simplex Unscented Kalman Filter (SSUKF) based on the Spherical Simplex Unscented Transformation [23] to estimate the mean equinoctial elements directly from the tracking data.

ACKNOWLEDGEMENT

APPENDIX A – GPS Input File

```
CONTROL   DC                                              GPSSAT     0099999
EPOCH               1081109.0          001446.0000
ELEMENT1 10  1  1-1.36990181020000E+04-8.48585960000000E+03+2.14410746720000E+04
ELEMENT2          +9.77264236179275E-01-2.46212135521801E+00-3.53083683724679E-01
OBSINPUT 20         1081109001446.0000  1081111001446.0000
ORBTYPE   5  1 11  43200.0000          1.0
DMOPT
OBSDEV   21 22 23  10.0                10.0                         10.0
END
DCOPT
CONVERG  30  3     0.0001
EDIT         3     3.0
PRINTOUT  1      4   10.0
END
OGOPT
POTFIELD  1 11
SSTESTFL  1  2  0  0.0
SSTAPGFL  1  0  0  0.0                 0.0                 1.0
SPGRVFRC  1  1  1  2.0                 1.0                 1.0
SPTESSLC  6  6  4  2.0                -8.0                 8.0
SPNUMGRV  7  1 10  2.0                 2.0                 3600.0
SPZONALS  8  5 11
SPJ2MDLY  4  4  5  2.0
SPMDAILY  8  8  5
POLAR     1        1.0
MAXDEGEQ  1        16.0
MAXORDEQ  1        16.0
SOLRAD    1        1.0
SOLRDPAR  1
```

```
NCBODY    1  2  3
SCPARAM             +1.00000000000000E-06+1.00000000000000E+02
STATEPAR  3  1
STATETAB  1  2  3  4.0               5.0                 6.0
END
FIN
```

APPENDIX B – EGP Input File

```
CONTROL    DC                                          SATSAT-0   0016908
EPOCH              1020829.0          215305.292
ELEMENT1  1  6  1 7866.628863685799  1.51219480042467E-03 49.99690802918278
ELEMENT2  0       234.0990114753167  44.20804018942176   315.9207248379734
OBSINPUT  5        1020829215305.292  1020908215305.292
ORBTYPE   5  1 11  43200.0           1.0
DMOPT
/L75L  1 0999  3      75.8889        0505202.5723        0002010.0469
/L79L  1 0999  3     804.5133        -351858.1309        1490035.5633
/L11L  1 0999  3    1839.4914        0325330.2604        2433438.3848
/L23L  1 0999  3     274.7100        0434725.8481        1252636.4466
/L68L  1 0999  3     760.5880        0245437.9659        0462401.3133
/L73L  1 0999  3     102.1025        0333439.6990        1355613.3396
/L66L  1 0999  3      98.7288        0362754.9151        3534740.8834
/L74L  1 0999  3     539.8719        0470401.6902        0152936.0992
/L06L  1 0999  3     241.8078        -290247.3956        1152048.2860
/L04L  1 0999  3    2004.7519        0304048.9635        2555905.2899
/L72L  1 0999  3      28.3126        0310551.1457        1211130.2640
/L94L  1 0999  3      31.8214        0565654.7843        0240332.6660
/L62L  1 0999  3      74.5050        0601301.7555        0242340.3816
/L38L  1 0999  3    1407.2622        -255322.9546        0274110.2274
/L09L  1 0999  3      19.6660        0390114.1792        2831020.3022
/L93L  1 0999  3     665.8497        0490839.9041        0125240.8289
/L36L  1 0999  3    2489.3114        -162756.5816        2883025.3609
/L20L  1 0999  3    3067.9800        0204225.9865        2034438.7206
/L26L  1 0999  3      82.1399        0393624.9678        1155331.3934
/L70L  1 0999  3    1323.3081        0434516.8903        0065516.0429
/L63L  1 0999  3     951.8148        0465238.0256        0072754.7913
END
DCOPT
CONVERG  30  3  1  0.0001
EDIT          3    3.0
BATCHTYP  7
TRACKELV  3        1.0
TRACKELV 13        3.0
/L75L    001       2
/L79L    001       2
/L11L    001       2
/L23L    001       2
/L68L    001       2
/L73L    001       2
/L66L    001       2
/L74L    001       2
/L06L    001       2
/L04L    001       2
/L72L    001       2
/L94L    001       2
/L62L    001       2
/L38L    001       2
/L09L    001       2
/L93L    001       2
/L36L    001       2
/L20L    001       2
```

```
/L26L    001         2
/L70L    001         2
/L63L    001         2
PRINTOUT  1    4   10.0
ELLMODEL  1         6378.13655         298.256421867
END


OGOPT
DRAG      1         1.0
ATMOSDEN       1
DRAGPAR   2         2.0                                      1.0
DRAGPAR2  1  1
POTFIELD  1 11
STATEPAR  3  1
STATETAB  1  2  3   4.0              5.0              6.0
SETIDE    1         0.29D0
SOLRAD    1         1.0
SOLRDPAR  2         1.2                                      0.005
POLAR     1         1.0
SPGRVFRC  1  1  1   1.0              1.0              1.0
SSTESTFL  1  2  0   0.0
SSTAPGFL  1  0  0   1.0              6.0              1.0
SPTESSLC 14 14  4   2.0             -10.0             10.0
SPZONALS  8  7 16
SPMDAILY 14 14 12
SPJ2MDLY  8  8  6   1.0
AVRDRAG   5  3  3
RESONPRD            259200.0
MAXDEGEQ  1         30.0
MAXORDEQ  1         30.0
NCBODY    1  2  3
SCPARAM            +3.14150000000000E-06+6.85000000000000E+02
END
FIN
```

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