ORBIT DETERMINATION AND DATA FUSION IN GEO

Joshua T. Horwood, Aubrey B. Poore

Numerica Corporation, 4850 Hahns Peak Drive, Suite 200, Loveland CO, 80538

Kyle T. Alfriend

Department of Aerospace Engineering, Texas A&M University, College Station, TX 77843

ABSTRACT

This paper proposes a simple yet reliable method for performing orbit determination or track initiation in the geosynchronous (GEO) regime of space using angle-only sensor observations. The main discovery communicated in this paper is a new online metric which allows the operator to assess in realtime if there is adequate observational data to solve the orbit determination problem and hence initiate a robust track state estimate and consistent covariance.

1. INTRODUCTION

A statistically rigorous treatment of uncertainty in the space surveillance network is a prerequisite to the robust tracking and reconnaissance of resident space objects (RSOs) and to supporting space situational awareness (SSA) operations such as conjunction analysis, sensor resource management, and anomaly (e.g., maneuver, change) detection. Common amongst these problems is the requirement of orbit determination which necessitates (i) the initiation of a track state from a sequence of sensor observations and (ii) a proper characterization of the uncertainty in the estimate often in the form of a Gaussian covariance matrix. This latter requirement is called *covariance consistency*.[1]

The orbit determination problem in the geosynchronous (GEO) regime of space is of particular interest due to the ubiquity of angle-only electro-optical (EO) sensor observations and the challenges arising from a lack of observability in some RSO state components. This problem is most difficult when considering angle-only measurements from a *single sensor* because often there is high accuracy along the line-of-sight but little information in the observer-to-RSO range and velocity. Such issues in GEO are well known and have been addressed by many researchers in the field. The purpose of this paper is to provide some fresh insight into the problem and to suggest means for improvement. In particular, the paper addresses the dependency of RSO state covariance consistency on the quantity, time diversity, and geometric diversity of angle-only observations from a single EO sensor. Further, a metric is proposed which allows the operator to assess *online*, in the absence of "truth," if there is sufficient data to robustly solve the orbit determination problem and achieve covariance consistency.

The proposed method for initiating a six-dimensional (6D) RSO track state (in either Cartesian Earth-centered-inertial or orbital element coordinates) and a consistent measure of its uncertainty (in the form of a full rank covariance matrix) takes a two-pronged approach in GEO. Firstly, single EO sensor angle-only measurements over a *single* sensor pass are fused into one single, smoothed, composite measurement and covariance by solving a *linear* batch least squares problem. This composite measurement, which we call a *GEO tracklet*, takes the form of a four-dimensional sensor observation encompassing both angles (e.g., right ascension and declination) and their rates. Secondly, GEO tracklets (hypothesized to emanate from a common RSO) over *multiple* sensor passes are fused into a full 6D track state and covariance by solving another batch estimation problem. The ability to generate a fully determined 6D state and consistent covariance in the second step is contingent on the time and geometric diversity of the GEO tracklets. Finally, once a full 6D RSO state and covariance is established through this method, traditional sequential filtering algorithms can be used (e.g., the extended or unscented Kalman filters) for subsequent space catalog maintenance and other supporting SSA operations.

Additionally, parametric studies on the time and geometric diversity of the sensor observations suggest that the *condition number of the correlation coefficient matrix* (derived from the RSO state covariance) provides a metric to assess the Gaussian assumption and consistency of the state uncertainty. Thus, the metric allows the operator to delay the initiation of a full 6D state and covariance (rather than risk misrepresenting the uncertainty by an inconsistent covariance) until the diversity improves or a composite measurement of the same RSO from a second sensor becomes available.

The plan of the paper is as follows. In Section 2, we review the general mathematical framework of orbit determination and the resulting nonlinear least squares problem. In Section 3, we specialize this framework to GEO and develop our proposed two-step approach to orbit determination using angle-only data. The online metric for assessing covariance consistency is defined in Section 4. Finally, results are presented in Section 5 and conclusions are made in Section 6.

2. ORBIT DETERMINATION AND BATCH ESTIMATION

In this section, we review the general mathematical framework for orbit determination and batch estimation both with and without process noise in the dynamics. In the next section, we specialize the framework to GEO and propose a practical algorithm for initiating and fusing tracks in GEO from angle-only data from multiple EO sensors. In particular, we show how uncertainty (covariance) consistency in the initiated track state can be achieved and tested online despite not having any range data in the observations.

Given a sequence of m measurements or sensor observations $\boldsymbol{Z}_m \equiv \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m\}$ at times $t_1, \ldots, t_m$, hypothesized to emanate from a common object, the objective of orbit determination and batch estimation is to obtain a representation of the (joint) posterior probability density function (PDF) $p(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m | \boldsymbol{Z}_m)$, where $\boldsymbol{x}_k$ denotes the dynamical state of the system at time t_k, and to extract meaningful statistics (e.g., mean, covariance) from it in a consistent manner. It is assumed that the state evolves according to the discrete-time model

$$\boldsymbol{x}_k = \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) + \boldsymbol{w}_{k-1}, \tag{1}$$

where $\{\boldsymbol{w}_k\}$ is a zero-mean white noise sequence with $\mathrm{E}[\boldsymbol{w}_k\, \boldsymbol{w}_j^T] = \mathbf{Q}_k\, \delta_{kj}$. The measurements $\boldsymbol{z}_k$ are related to the corresponding kinematic states $\boldsymbol{x}_k$ according to the discrete-time measurement model

$$\boldsymbol{z}_k = \boldsymbol{h}_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, \tag{2}$$

where $\{\boldsymbol{v}_k\}$ is a zero-mean white noise sequence with $\mathrm{E}[\boldsymbol{v}_k\, \boldsymbol{v}_j^T] = \mathbf{R}_k\, \delta_{kj}$. The measurement functions $\boldsymbol{h}_k$ are typically comprised of coordinate transformations from state space to sensor space. It is assumed in the model (2) that all sensors have been calibrated and all residual biases have been incorporated into the measurement noise sequence. The following independence assumptions are implied between the prior $\boldsymbol{x}_0$, the measurement noise sequence $\{\boldsymbol{v}_k\}$, and the process noise sequence $\{\boldsymbol{w}_k\}$:

$$\mathrm{E}[\boldsymbol{x}_0\, \boldsymbol{w}_k^T] = 0, \quad \mathrm{E}[\boldsymbol{x}_0\, \boldsymbol{v}_k^T] = 0, \quad \mathrm{E}[\boldsymbol{v}_k\, \boldsymbol{w}_j^T] = 0.$$

Appealing to Bayes' rule and the above assumptions, the joint posterior PDF is derived in Jazwinski[2] and is found to be

$$p(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m | \boldsymbol{Z}_m) = c\, p_0(\boldsymbol{x}_0) \prod_{k=1}^{m} p_{w_{k-1}}(\boldsymbol{x}_k - \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})) \prod_{k=1}^{m} p_{v_k}(\boldsymbol{z}_k - \boldsymbol{h}_k(\boldsymbol{x}_k)), \tag{3}$$

where c is a normalizing constant, and p_0 is the prior PDF of the state $\boldsymbol{x}_0$ at time t_0. Further, in (3), the p_{w_k} and p_{v_k}, for $k = 1, \ldots, m$, are the respective PDFs of the process and measurement noise processes. In practice, they are often assumed to be Gaussian with zero mean and covariances of $\mathbf{Q}_k$ and $\mathbf{R}_k$, respectively.

The posterior PDF (3) *is* the complete description of the uncertainty of the state at each of the measurement times. In practice, a finite dimensional representation of the uncertainty is sought. Thus, the emphasis of the batch estimation problem is on how statistical information can be extracted from (3) consistently and accurately. Nonlinear optimization theory provides a framework for computing the *modal trajectory* or *maximum a posteriori (MAP)* estimate of (3). For a Gaussian prior with $\boldsymbol{x}_0 \sim N(\bar{\boldsymbol{x}}_0, \bar{\mathbf{P}}_0)$ and Gaussian noise processes as described above, the modal trajectory is obtained by solving the least squares or batch problem

$$
\begin{aligned}
(\hat{\boldsymbol{x}}_0, \ldots, \hat{\boldsymbol{x}}_m)^{MAP} &= \underset{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_m}{\text{Maximize}}\ p(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m | \boldsymbol{Z}_m) \\
&= \underset{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_m}{\text{Minimize}}\ \frac{1}{2}\|\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0\|^2_{\bar{\mathbf{P}}_0^{-1}} + \frac{1}{2}\sum_{k=1}^{m}\|\boldsymbol{x}_k - \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})\|^2_{\mathbf{Q}_{k-1}^{-1}} + \frac{1}{2}\sum_{k=1}^{m}\|\boldsymbol{z}_k - \boldsymbol{h}_k(\boldsymbol{x}_k)\|^2_{\mathbf{R}_k^{-1}}.
\end{aligned}
\tag{4}
$$

In initial orbit determination, we do not have a prior; the term $\frac{1}{2}\|\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0\|^2_{\bar{\mathbf{P}}_0^{-1}}$ is removed from the above formulation. Further, in space surveillance applications, the process noise term is very small and is commonly discarded (at least over the short time frames over which orbit determination is performed). In other words, deterministic dynamics are assumed which are typically specified in the form of an ordinary differential equation

$$\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}(t), t).$$

The continuous-time dynamics implied by the above equations of motion can be cast in discrete-form using the solution flow,[3] i.e., $\boldsymbol{x}(t) = \boldsymbol{\phi}(t; \boldsymbol{x}_0, t_0)$, so that $\boldsymbol{x}_k = \boldsymbol{\phi}(t_k; \boldsymbol{x}_{k-1}, t_{k-1}) \equiv \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})$. With deterministic dynamics and no prior, the MAP estimate at (the final measurement time) t_m simplifies to

$$
\hat{\boldsymbol{x}}_m^{MAP} = \underset{\boldsymbol{x}_m}{\text{Maximize}}\ p(\boldsymbol{x}_m|\boldsymbol{Z}_m) = \underset{\boldsymbol{x}_m}{\text{Minimize}}\ \frac{1}{2}\sum_{k=1}^{m}\left\|\boldsymbol{z}_k - \boldsymbol{h}_k\big(\boldsymbol{\phi}(t_k; \boldsymbol{x}_m, t_m)\big)\right\|^2_{\mathbf{R}_k^{-1}}.
\tag{5}
$$

Methods for solving nonlinear least squares problems, such as Gauss-Newton, full Newton, and quasi-Newton updates, along with globalization methods such as line search and trust region methods including Levenberg-Marquardt,[4] are efficient and mature and will not be discussed further here. In the astrodynamics community, such solution techniques are called *differential correction* methods. In any nonlinear least squares problem such as (4) or (5), one must provide the solver a starting guess in order to initiate the differential correction method. This is the *initial orbit determination (IOD)* problem. In the case of measurement data from a single radar or EO sensor, a first estimate can be obtained using the classical methods of Lambert or Gauss (see, for example, Vallado[5]). Additionally, for angle-only observations, a recent algorithm due to Gooding[6] has shown promise for IOD scenarios involving both ground-based and space-based EO sensors.[7]

Once the MAP estimate (5) has been found by solving a nonlinear least squares problem, the (osculating) covariance of the state at time t_m can be computed from[8]

$$
\hat{\mathbf{P}}_m = \left[\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}^T\left(\frac{1}{2}\sum_{k=1}^{m}\left\|\boldsymbol{z}_k - \boldsymbol{h}_k\big(\boldsymbol{\phi}(t_k; \boldsymbol{x}, t_m)\big)\right\|^2_{\mathbf{R}_k^{-1}}\right)\right]^{-1}_{\boldsymbol{x}=\hat{\boldsymbol{x}}_m^{MAP}},
\tag{6}
$$

where $\nabla_{\boldsymbol{x}}$ is the gradient with respect to $\boldsymbol{x}$ viewed as a column operator. The MAP estimate (5) and covariance (6) provide a Gaussian approximation of the posterior PDF. We remark that (6) is precisely the Cramér-Rao lower bound[9] which is the inverse of the Fisher information. Performing a singular value decomposition (SVD)[10] of the Fisher information reveals the amount of information along each observable direction. The SVD analysis is of particular importance when considering angle-only measurements of objects in GEO because often there is high accuracy along the line-of-sight but little information in the observer-to-RSO range and velocity.

3. SPECIALIZATIONS TO GEO

The primary difficulty of the orbit determination problem in GEO is the absence of range data in the corresponding sensor observations. Although a well-defined solution exists to the nonlinear least squares problem (5) when processing angle-only data over a single sensor pass (the nonlinearities in the dynamics guarantee mathematical observability), it is often the case that the estimated range is unrealistic (i.e., outside the near-Earth environment) and the uncertainty in range is so large rendering any such solution of little practical use. In such cases, the instantiation of a robust track state and consistent covariance is simply not achievable.

In what follows, we propose a simple yet effective two-step method for solving the orbit determination problem in GEO which first processes angle-only data from a single sensor pass to generate a smoothed, composite measurement called a *GEO tracklet* and then fuses multiple GEO tracklets (over multiple sensor passes) into a 6D state and covariance. An online metric, defined in the next section, can then detect if there is sufficient data to robustly solve the orbit determination problem and if the estimated covariance correctly characterizes the uncertainty in the track state estimate.

Step 1: Initiate GEO Tracklets

When processing angle-only data over a single sensor pass in GEO, the limited time and geometric diversity of the observations often prohibits an accurate range (and range-rate) estimate of the object. Further, the nonlinear least squares problem (5) is highly ill-conditioned and the resulting covariance (6) lacks consistency. Thus, in this initial processing stage, *we propose to not even attempt to estimate the range and range-rate* and instead compute a smoothed observation formed from the angle and angle-rate information. Specifically, over the short time spans of a single sensor pass, a *constant acceleration model in (angle) measurement space* is a good approximation to the (nonlinear) dynamics. Equivalently, given two-dimensional angle-only (e.g., right ascension, declination) observations $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m$, we fit a quadratic polynomial through the observations by solving*

$$\underset{\boldsymbol{\theta}_q, \dot{\boldsymbol{\theta}}_q, \ddot{\boldsymbol{\theta}}_q}{\text{Minimize}} \;\; \frac{1}{2} \sum_{k=1}^{m} \left\| \boldsymbol{z}_k - \boldsymbol{\theta}_q - (t_k - t_q)\dot{\boldsymbol{\theta}}_q - \tfrac{1}{2}(t_k - t_q)^2 \ddot{\boldsymbol{\theta}}_q \right\|^2_{\mathbf{R}_k^{-1}}. \tag{7}$$

The optimization problem (7) is *computationally cheap* because it is only a *linear* least squares problem. A *GEO tracklet* is the four-dimensional (4D) "composite measurement" formed from the angle $\boldsymbol{\theta}_q$ and angle rate $\dot{\boldsymbol{\theta}}_q$ estimates along with the corresponding 4×4 covariance matrix. We acknowledge that a similar idea of using basic kinematics to precisely estimate the angular and angular-rate information was also proposed by Maruskin *et al.*[11]

Step 2: Fuse GEO Tracklets

Let $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m$ be a sequence of 4D GEO tracklets with corresponding covariances $\mathbf{R}_1, \ldots, \mathbf{R}_m$, possibly originating from different sensors and hypothesized to emanate from the same object. By considering observations over multiple sensor passes in this step, the data often has adequate time and geometric diversity to permit an accurate estimate of the range and range-rate of the object and hence the full 6D track state. The fully determined 6D state $\boldsymbol{x}_m$ and covariance (in either Cartesian Earth-centered-inertial coordinates or orbital elements) is computed by solving the nonlinear least squares batch problem (5). Because the time separation between the GEO tracklet data could be long, the dynamics encoded by the solution flow $\boldsymbol{\phi}$ in (5) should encapsulate a sufficiently high-order gravity model (a J_2 model at the very least). Once a fully determined track state and uncertainty is established through these two steps, one can then apply standard methods for nonlinear filtering (e.g., the extended or unscented Kalman filters) to update the track as new information becomes available.

4. AN ONLINE METRIC FOR COVARIANCE CONSISTENCY

Just how much time and geometric diversity and the number of GEO tracklets needed to achieve a precise 6D state estimate and consistent covariance is a tricky problem especially if such an assessment needs to be done online in the absence of truth data. Parametric studies conducted in this research and outlined in the next section suggest a strong correlation between covariance consistency and the condition number of the correlation coefficient matrix of the state estimate. Given the components P_{ij} of the covariance matrix $\mathbf{P}$ of the state estimate, the components of the correlation coefficient matrix are defined by

$$\rho_{ij} = \frac{P_{ij}}{\sqrt{P_{ii} P_{jj}}}. \tag{8}$$

The condition number of a matrix is the ratio of its largest singular value to its smallest singular value.[10] The condition number of $\boldsymbol{\rho}$ is a dimensionless quantity which is independent of choice of units. Although merely

*The index q in (7) usually coincides with the index of middle observation; i.e., $q = \lfloor m/2 \rfloor$.

Table 1. GEO tracklet initiation. Tabulated numbers are the p-values corresponding to the null hypothesis that the computed Mahalanobis distances come from a $\chi^2(4)$ distribution.

	$N_{obs} = 4$	$N_{obs} = 40$	$N_{obs} = 80$	$N_{obs} = 200$	$N_{obs} = 400$
GEO_1	0.970	0.998	0.996	0.004	0.
GEO_2	0.015	0.700	0.490	0.520	0.
GEO_3	0.520	0.020	0.340	0.180	0.
LEO_1	0.	0.	0.	0.	0.

a heuristic justified by parametric studies, *a large condition number of* $\boldsymbol{\rho}$ *can imply a lack of covariance consistency in the state estimate*. An alternate analysis could consider the higher-order cumulants of the posterior PDF and then verify that they are sufficiently small to justify the Gaussian approximation (see, for example, Horwood *et al.*[8]). Notwithstanding this comment, the condition number of a matrix is cheap to compute relative to higher-order cumulants of a PDF. Further, if the condition number of $\boldsymbol{\rho}$ is above some threshold, one can delay the fusion of the GEO tracklet sequence until more information becomes available rather than risk initiating an inconsistent measure of the state uncertainty. Exactly what this threshold should be is analyzed in the next section.

5. RESULTS

Validation of the proposed orbit determination method in GEO developed in the previous two sections uses simulated data representative of a real dataset. In generating the results of this section, simulated right ascension, declination observations are taken (from a truth trajectory) every 6 s with errors of $2''$. Four truth objects are considered. The truth objects labeled GEO_1, GEO_2, and GEO_3 have elevation angles of $50°$, $30°$, and $15°$, respectively, relative to an Earth-fixed EO sensor. The LEO_1 truth object has a nearly circular orbit with altitude of 640 km, inclination of $30°$, and elevation angle of $18°$ when viewed from the fixed sensor[†].

Table 1 shows the results of solving the linear least squares problem (7) for GEO tracklet initiation. In the table we vary the truth object and the number N_{obs} of right ascension, declination observations used to generate the GEO tracklet (or, equivalently, the "length" of the tracklet). The tabulated numbers are the p-values corresponding to the null hypothesis that the Mahalanobis distance

$$\mathcal{M} = (\boldsymbol{x}_{est} - \boldsymbol{x}_{truth})^T \mathbf{P}_{\boldsymbol{x}}^{-1} (\boldsymbol{x}_{est} - \boldsymbol{x}_{truth})$$

comes from a χ^2-distribution with four degrees of freedom[‡]. All numbers in the table (and those in Table 2) are based on data taken over 1000 Monte-Carlo runs. For the three GEO objects, a consistent estimate of the right ascension, declination, and their rates, along with a corresponding 4×4 covariance matrix (by definition, this is a GEO tracklet), can be established (since p-values are generally larger than 0.01) for short to medium length tracks ($\lesssim 20$ minutes or $N_{obs} \lesssim 200$). For long tracks[§] ($\gtrsim 20$ minutes or $N_{obs} \gtrsim 200$), covariance consistency breaks down. This is unsurprising because the simplified assumption that the dynamics can be approximated by a constant acceleration model in sensor space becomes less valid as the time span of the observations increases. This assumption also breaks down in LEO irregardless of track length. In these situations, it is necessary to solve the full nonlinear least squares problem (5), as one commonly does when performing orbit determination with radar observations.

Table 2 shows the results of fusing multiple 4D GEO tracklets by solving the batch least squares problem (5) in equinoctial orbital element space[12] using a degree and order 70 gravity model. Tabulated numbers in black are the p-values corresponding to the null hypothesis that the computed Mahalanobis distances

[†]The elevation angles (for both the GEO and LEO truth objects) correspond to the first observation. For short tracks, the elevation angles will be approximately constant over the duration of the track.

[‡]In statistical hypothesis testing, one often rejects the null hypothesis if the p-value is typically less than 0.05 (or 0.01), corresponding to a 5% (1%) chance of observing results at least that extreme, given a true null hypothesis.

[§]Such long tracks, especially in LEO, are generally not in view over their entire duration. The purpose of their inclusion was purely mathematical in order to "stress-test" the simplified assumption of the constant acceleration model in sensor space.

Table 2. Fusion of multiple GEO tracklets from a single sensor. Tabulated numbers in **black** are the p-values corresponding to the null hypothesis that the computed Mahalanobis distances come from a $\chi^2(6)$ distribution. Tabulated numbers in **blue** are the condition numbers of the correlation coefficient matrix.

Time separation between tracklets (in orbital periods)	$N_{tracklets} = 2$	$N_{tracklets} = 3$	$N_{tracklets} = 4$	$N_{tracklets} = 8$	$N_{tracklets} = 16$
0.01	0.	0.	0.	0.	0.1239
	$7.09 \cdot 10^{21}$	$2.68 \cdot 10^{7}$	$1.06 \cdot 10^{7}$	$2.07 \cdot 10^{6}$	$2.51 \cdot 10^{5}$
0.05	0.	0.	0.6456	0.9158	0.2559
	$4.28 \cdot 10^{14}$	$8.48 \cdot 10^{5}$	$1.71 \cdot 10^{5}$	$4.90 \cdot 10^{3}$	$1.64 \cdot 10^{2}$
0.1	0.	0.4907	0.0001	0.1139	0.2390
	$8.30 \cdot 10^{9}$	$1.59 \cdot 10^{5}$	$9.41 \cdot 10^{3}$	$1.64 \cdot 10^{2}$	$1.42 \cdot 10^{1}$
0.2	0.	0.6901	0.9953	0.9675	0.8995
	$1.71 \cdot 10^{9}$	$1.61 \cdot 10^{4}$	$2.59 \cdot 10^{2}$	$1.48 \cdot 10^{1}$	$1.36 \cdot 10^{2}$
0.3	0.	0.9896	0.4679	0.1269	0.9034
	$6.50 \cdot 10^{8}$	$1.01 \cdot 10^{4}$	$2.01 \cdot 10^{1}$	$1.17 \cdot 10^{1}$	$1.31 \cdot 10^{1}$
0.4	0.	0.8640	0.6012	0.7757	0.2107
	$2.76 \cdot 10^{8}$	$2.50 \cdot 10^{4}$	$1.10 \cdot 10^{1}$	$1.35 \cdot 10^{1}$	$1.27 \cdot 10^{1}$
0.5	0.	0.1153	0.0690	0.9530	0.9139
	$1.94 \cdot 10^{5}$	$2.85 \cdot 10^{4}$	$2.02 \cdot 10^{4}$	$1.90 \cdot 10^{4}$	$1.88 \cdot 10^{4}$
0.6	0.	0.3301	0.1468	0.8665	0.6975
	$8.63 \cdot 10^{7}$	$3.97 \cdot 10^{4}$	$1.10 \cdot 10^{1}$	$1.35 \cdot 10^{1}$	$1.30 \cdot 10^{1}$
1	0.	0.	0.	0.	0.
	$3.31 \cdot 10^{12}$	$1.36 \cdot 10^{12}$	$7.67 \cdot 10^{11}$	$9.92 \cdot 10^{10}$	$3.42 \cdot 10^{10}$

(formed from the 6D track state and covariance) come from a χ^2-distribution with six degrees of freedom. The numbers in blue are the corresponding condition numbers of the correlation coefficient matrix (8). The first general observation we make is that covariance consistency cannot be obtained by fusing just two GEO tracklets even if their time separation ranges from a few minutes to a complete orbital period (because the p-values are all infinitesimally small). However, if attempting to fuse three or more tracklets, one can generally initiate a consistent covariance (as most of the p-values are greater than 0.05) although there are some exceptions. In particular, covariance consistency might not be achievable if there is a lack of time diversity. For the fusion of three tracklets or more, a time separation of at least 0.1 orbital periods or about 2.4 hours between tracklets is needed[¶]. Additionally, a lack of geometric diversity can prohibit covariance consistency. In particular, if the time separation between tracklets is one complete orbital period, a consistent state and covariance cannot be initiated. Understandably, observability issues arise in this case since the object is seen at essentially the same location in the sky when the sensor observations are made. We further remark that the underlying conclusions gathered from Table 2 do not change when initiating a track state-covariance pair in Cartesian Earth-centered inertial coordinates rather than in equinoctial orbital elements.

Finally, the results in Table 2 suggest that the condition number of the correlation coefficient matrix $\boldsymbol{\rho}$ can provide an online test for covariance consistency (at least in the GEO regime). Indeed, all scenarios in which covariance consistency was obtained (with respective p-values greater than 0.05) had $\text{cond}(\boldsymbol{\rho}) \lesssim 10^5$. Therefore, if the condition number is greater than $\sim 10^5$, it is advised not to initiate a 6D track state and covariance (at the risk of misrepresenting the uncertainty by an inconsistent covariance) and instead wait until additional observations or GEO tracklets become available.

6. CONCLUSIONS

In this paper, we reviewed the general framework of the batch estimation problem for track initiation, both with and without process noise, and showed how a robust solution could be obtained for the problem in GEO. Further, we developed a *reliable online metric*, based on the condition number of the correlation coefficient

[¶]In Table 2, the p-value of 0.0001 obtained for the 4 tracklet, 0.1 orbital period separation case appears to be an exception. Upon a more careful analysis of the data, there was one Monte-Carlo run which produced an outlier resulting in this small p-value. Rerunning this scenario with 10,000 Monte-Carlo runs (instead of only 1000) produced a statistically insignificant p-value of 0.5587.

matrix $\boldsymbol{\rho}$, to assess if there is sufficient data to initiate a robust track state and consistent covariance. Based on the Monte-Carlo simulations conducted in this paper, a simple rule of thumb for initiating a good orbit in GEO is the availability of at least 3 GEO tracklets and $\mathrm{cond}(\boldsymbol{\rho}) < 10^5$.

ACKNOWLEDGMENTS

This work was funded by the Air Force Office of Scientific Research through a research grant (FA9550-08-1-0419) and by a Phase I SBIR from the Air Force Research Laboratory, Rome NY (FA8750-11-C-0147).

REFERENCES

[1] O. E. Drummond, A. J. Perrella, Jr., and S. Waugh, "On target track covariance consistency," vol. 6236, p. 623615, SPIE, 2006.

[2] A. H. Jazwinski, *Stochastic Processes and Filtering Theory.* New York: Dover, 1970.

[3] R. S. Park and D. J. Scheeres, "Nonlinear mapping of Gaussian statistics: theory and applications to spacecraft trajectory design," *Journal of Guidance, Control, and Dynamics*, vol. 29, no. 6, pp. 1367–1375, 2006.

[4] R. Fletcher, *Practical methods of optimization.* Chichester: John Wiley & Sons, 1991.

[5] D. A. Vallado, *Fundamentals of Astrodynamics and Applications.* New York: Springer, 2007.

[6] R. H. Gooding, "A new procedure for the solution for the classical problem of minimal orbit determination from three lines of sight," *Celestial Mechanics and Dynamical Astronomy*, vol. 66, no. 1, pp. 387–423, 1997.

[7] D. A. Vallado, "Evaluating Gooding angles-only orbit determination of space based space surveillance measurements," in *Proceedings of the AAS George Born Astrodynamics Symposium*, (Boulder, CO), May 2010.

[8] J. T. Horwood, N. D. Aragon, and A. B. Poore, "Covariance consistency for track initiation using Gauss-Hermite quadrature," in *SPIE Proceedings: Signal and Data Processing of Small Targets 2010*, vol. 7698, 2010.

[9] Y. Bar-Shalom, X.-R. Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation.* New York: John Wiley & Sons, 2001.

[10] G. Golub and C. Van Loan, *Matrix Computations.* Baltimore, MD: John Hopkins University Press, 1996.

[11] J. M. Maruskin, D. J. Scheeres, and K. T. Alfriend, "Correlation of optical observations of objects in Earth orbit," *Journal of Guidance, Control, and Dynamics*, vol. 32, no. 1, pp. 194–209, 2009.

[12] R. A. Broucke and P. J. Cefola, "On the equinoctial orbit elements," *Celestial Mechanics*, vol. 5, pp. 303–310, 1972.