# On-Orbit Range Set Applications 

Marcus J. Holzinger<br>Aerospace Engineering Department<br>Texas A $\mathcal{B} M$ University

Daniel J. Scheeres<br>Aerospace Engineering Sciences<br>University of Colorado at Boulder


#### Abstract

History and methodology of $\Delta v$ range set computation is briefly reviewed, followed by a short summary of the $\Delta v$ optimal spacecraft servicing problem literature. Service vehicle placement is approached from a $\Delta v$ range set viewpoint, providing a framework under which the analysis becomes quite geometric and intuitive. The optimal servicing tour design problem is shown to be a specific instantiation of the metric- Traveling Salesman Problem (TSP), which in general is an NP-hard problem. The $\Delta v$-TSP is argued to be quite similar to the Euclidean-TSP, for which approximate optimal solutions may be found in polynomial time. Applications of range sets are demonstrated using analytical and simulation results.


## 1 INTRODUCTION

Range sets, traditionally used in aircraft settings, define a set of reachable points given a fuel constraint and starting position(s). These sets are excellent mission planning and scenario evaluation tools, as they allow complex, nonlinear differential systems to be expressed as intuitive, end-user friendly geometric problems. Several spacecraft applications can benefit directly from the computation of time-independent, characteristic velocity $(\Delta v)$-constrained range sets. In particular, mission design activities and on-orbit service planning may be performed in a geometric, intuitive framework given orbit range sets. This paper is concerned with the application of the $\Delta v$ metric and range sets to the service vehicle placement and on-orbit service tour planning problems.

Range set computation for small, linearized regions about reference orbits has been studied intermittently for more than 40 years by both Breakwell [1] and Marec [2]. More recently, Xue et al. have examined freetime, single-impulse range sets [3] and Li et al. have examined reachable domain sets using two types of thrust [4]. It has also been shown that spacecraft $\Delta v$ may be viewed as a distance metric [5], and that using $\Delta v$ as a generalized independent parameter (GIP), a Generalied Metric Range Set (GMRS) representing timeindependent, $\Delta v$-constrained orbit range sets may be computed [6]. Using Gauss' Variational Equations (GVEs) with $\Delta v$ as the independent parameter, time-independent range sets for arbitrary orbit transfers under $J_{2}$ perturbations have been validated and the optimal control policy derived [7]. With the solution to the on-orbit range set computation under $J_{2}$ perturbations in hand, specific applications may be explored.

The problem of designing optimal refueling strategies (also known as 'refueling tours,' or simply 'tours') is a particular case of the Traveling Salesman Problem (TSP) that has been extensively studied for nearly 200 years, first considered by W. R. Hamilton and T. Kirkman, then formally defined in the 1930s by K. Menger [8]. Interestingly, because minimum- $\Delta v$ cost has been proven to be a properly defined distance metric (namely the triangle inequality property) [5], algorithms applying to the metric-TSP subproblem may be used [9]. Additionally, there are tour optimization algorithms specifically designed for the Euclidean distance metric that can approximate the optimal path with a cost multiple of $1+1 / c$ (for $c>0$ arbitrarily large) in polynomial time [10], rather than the exponential time of the general problem [9].

Supposing that these same algorithms may be applied to $\Delta v$ metric-TSP (here dubbed ' $\Delta v$-TSP') scenarios, realistic computation of $\Delta v$-TSP solutions appears to be entirely feasible. Euclidean-TSP solutions have been found for in excess of 85,000 nodes, far exceeding the current number of active on-orbit objects $(1,300)$, as well as all trackable objects with diameters greater than $10 \mathrm{~cm}(19,000)$ [11].

There are a number of past efforts examining the on-orbit optimal servicing tour problem. Alfriend et al. examined the servicing of Geosynchronous objects $[12,13]$ in which the analysis is restricted to spacecraft with circular orbits, identical periods, and small differential inclinations. Optimal servicing strategies between spacecraft in a circular orbit constellation have also been considered with minimum fuel use, equalized fuel use, and combinations thereof [14]. Further treatment of the same problem has also utilized network graph theory formalisms to aid analysis and propose new fuel expenditure and transfer strategies [15].

The application of range set computation to both service vehicle placement and optimal servicing tour scenario analysis is examined in this paper. The Background Theory section briefly reviews $\Delta v$-metrics, optimal control policy, and the computation of $\Delta v$-range sets. The utility of $\Delta v$ range set computation to both the sevice vehicle placement problem and the optimal service tour problem is then demonstrated in the Applications section. Specific contributions of this paper are a) demonstration of the utility of $\Delta v$ range sets to the problem of placing service vehicles on-orbit, and b) the observation that the minimum- $\Delta v$ servicing tour problem is a specific instantiation of not just the standard TSP, but the metric-TSP, and nearly identical to the Euclidean-TSP, for which well-developed algorithms may be applied.

## 2 BACKGROUND THEORY

Prior to discussing applications, there are several specific theoretical results serving as the foundation of $\Delta v$ range set theory that must be briefly introduced. First, the result demonstrating that minimum $\Delta v$ costs between initial and final states is a metric is stated. Second, the Generalized Independent Parameter (GIP) HJB PDE is given and discussed. Lastly, the combination of both $\Delta v$ metrics and the GIP HJB PDE is shown to provide a means by which optimal $\Delta v$ range sets may be computed in the presence of $J_{2}$ perturbations. Before discussing these items, several assumptions (initially outlined in [7]) are first made.
[A1] The user is unconcerned with the duration of an optimal maneuver from an initial orbit element set $\propto_{0}$ to a final orbit element set $\propto_{f}$. The problem is considered a 'free-time' optimal control problem.
[A2] The general accelerations $a_{r}, a_{\theta}$, and $a_{h}$ in the rotating Hill frame are the result of corresponding control accelerations $u_{r}, u_{\theta}$, and $u_{h}$.
[A3] For the first part of this analysis dynamics in the ascending node $\Omega$ are ignored (but are discussed in §2.4).
[A4] Similar to the ascending node $\Omega$, the argument of periapsis $(\omega)$ dynamics are ignored for the first portion of this analysis (but are discussed in §2.4).
[A5] The control inputs $\mathbf{u}$ are impulsive. Because of mapping ambiguities between time and $\Delta v$, and because under control accelerations the true anomaly $f$ changes, it is expedient to consider impulsive control inputs.
[A6] When integrating in the $\Delta v$ independent parameter space, it is very convenient to use the true anomaly $f$ as a control variable. This is perfectly sensible, as a spacecraft operator can use $f$ as a timing/phasing control variable. The parameter $f$ is considered a control parameter with $f \in[0,2 \pi)$.

### 2.1 Minimum $\Delta v$ Cost as a Metric

The following Theorem from [5] describing how performance indices for minimized Optimal Control Problems (OCPSs) is introduced, followed by a short discussion on how $\Delta v$ is also a distance metric.

## Theorem 2.1. Optimal Control Problem Distance

The function

$$
\begin{array}{r}
d_{O C P}(a, b)=\inf _{u \in U}\left[\int_{t_{a}}^{t_{b}} \mathcal{L}(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau), \tau) \mathrm{d} \tau\right] \\
\text { s.t. } \dot{\boldsymbol{x}}(t)=\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)  \tag{1}\\
\boldsymbol{h}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{t}) \leq \boldsymbol{O} \\
\boldsymbol{g}\left(\boldsymbol{x}_{a}, t_{a}, \boldsymbol{x}_{b}, t_{b}\right)=\boldsymbol{O}
\end{array}
$$

is a distance metric defined over the arguments a and b, where $\boldsymbol{x} \in \mathbb{R}^{n}, \boldsymbol{u} \in \mathbb{R}^{m}, t \in\left[t_{0}, t_{f}\right], \mathcal{L}: \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R} \rightarrow$ $\mathbb{R}, \mathcal{L}(\cdot, \cdot, \cdot) \geq 0 \forall \boldsymbol{x}, \boldsymbol{u}, t$ is the trajectory cost, $\boldsymbol{f}: \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R} \rightarrow \mathbb{R}^{n}$ describes the system dynamics, and $\boldsymbol{g}: \mathbb{R}^{n} \times \mathbb{R} \times \mathbb{R}^{n} \times \mathbb{R} \rightarrow \mathbb{R}^{r}$ is a function defining boundary conditions. The arguments $a=\left(\boldsymbol{x}_{a}, t_{a}\right)$ and $b=\left(\boldsymbol{x}_{b}, t_{b}\right)$, each defined on the cartesian product of the state space and time coordinates $\left(\mathbb{R}^{n} \times \mathbb{R}\right)$ must be members of the set of boundary values that satisfy the boundary condition equation $\boldsymbol{g}\left(\boldsymbol{x}_{a}, t_{a}, \boldsymbol{x}_{b}, t_{b}\right)=\boldsymbol{O}$, defined as

$$
(a, b) \in \mathcal{G} \equiv\left\{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \mid \boldsymbol{g}\left(\boldsymbol{x}_{\alpha}, t_{\alpha}, \boldsymbol{x}_{\beta}, t_{\beta}\right)=\boldsymbol{O}\right\}
$$

The boundary values a and b must also be members one another's reachability sets, defined as

$$
\begin{equation*}
b \in \mathcal{R}(U, \boldsymbol{h}, a) \equiv\left\{b=\left(\boldsymbol{x}_{b}, t_{b}\right) \mid\left(\boldsymbol{x}_{b}, t_{b}\right) \text { is reachable given } U, \boldsymbol{h}, \text { and } a\right\} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
a \in \mathcal{R}(U, \boldsymbol{h}, b) \equiv\left\{a=\left(\boldsymbol{x}_{a}, t_{a}\right) \mid\left(\boldsymbol{x}_{a}, t_{a}\right) \text { is reachable given } U, \boldsymbol{h}, \text { and } b\right\} \tag{3}
\end{equation*}
$$

The set $\mathcal{R}(U, \boldsymbol{h}, a)$ is the reachability set relative to $a$, and is a function of the allowable control set $U$ and trajectory inequality constraints $\boldsymbol{h}$ (see [16] for a brief optimal controls reachability introduction). Using this definition, given $(a, b) \in \mathcal{G}$, it is required that $b \in \mathcal{R}(U, \boldsymbol{h}, a)$ or $a \in \mathcal{R}(U, \boldsymbol{h}, b)$. A shorthand notation $a, b \in \mathcal{G}, \mathcal{R}$ is used to represent both of these cases.

## Proof:

The proof for Theorem 2.1 is given in detail in [5].
A common performance index used to compute $\Delta v$ cost is the $L_{1}$ norm of the input acceleration $\mathbf{u}(t)$ [17], defined as

$$
\begin{equation*}
\Delta v=d_{\Delta v}(a, b)=\int_{t_{0}}^{t_{f}}\|\mathbf{u}(\tau)\| \mathrm{d} \tau \tag{4}
\end{equation*}
$$

Through a straightforward application of Theorem 2.1, it is clear that using (4) as the performance index of choice also defines a metric distance in terms of $\Delta v$ between any two orbits and epochs $a=\left(\propto_{0}, t_{0}\right)$ and $b=\left(\propto_{f}, t_{f}\right)$.

### 2.2 Generalized Metric Range Sets

$\Delta v$ range sets have very special properties which are now briefly outlined. The Generalized Independent Parameter (GIP) HJB PDE derived in [6] is given as

$$
\begin{equation*}
\frac{\partial \tilde{V}}{\partial s}+\underset{\mathbf{u} \in U}{ }\left[\frac{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}+\frac{\partial \tilde{V}^{T}}{\partial \mathbf{x}} \frac{\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}\right]=0 \tag{5}
\end{equation*}
$$

where $s$ is a general independent parameter, $\tilde{V}$ is the value function, and $\tilde{l}$ is the slope of the mapping function from time $t$ to $s$. As discussed in detail in [6], if the independent parameter is also the performance index cost $(s=P)$ and $P$ is a metric, (5) reduces to

$$
\begin{equation*}
\frac{\partial \tilde{V}}{\partial s}+\underset{\mathbf{u} \in U}{\mathrm{opt}}\left[\frac{\partial \tilde{V}^{T}}{\partial \mathbf{x}} \frac{\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, s)}\right]=0 \tag{6}
\end{equation*}
$$

which has the same functional form as traditionally defined minimum time reachability problems $[18,19,20$, 21]. As such, existing numerical toolboxes may be used to propagate (6) and compute the zero-level sets, representing the boundaries of the range sets. For efforts discussed in this paper, I. M. Mitchell's toolbox is used [22].

## $2.3 \Delta v$ Range Set Computation

This section summarizes an extended, detailed derivation of the $\Delta v$ metric range computation given in [7]. Choosing to make the independent parameter in (5) $\Delta v$, as defined by (4), the independent parameter mapping function is chosen such that $l(\mathbf{x}(t), \mathbf{u}(t), t) \equiv\|\mathbf{u}(t)\|$. Applying Lemma 2.III from [6], choosing the state-space coordinates to be constants of motion (e.g. the classical orbit elements $\propto$, where $\dot{\boldsymbol{e}}=\mathbf{f}(\propto) \mathbf{u}$, shown in Fig. 1(a)) allows (6) to be well defined during periods in which $\|\mathbf{u}(t)\|=0$. Additionally, since a time-independent approach is now being considered (through the use of constants of motion as coordinates and per [A1]), the orbit true anomaly $f$ is also considered a control variable ([A6]). Reparameterizing the instantaneous control acceleration direction $\hat{\mathbf{u}}$ into Hill-frame azimuth and elevation coordinates (as shown in Fig. 1(b)), where $\hat{\mathbf{u}}$ is written as

$$
\hat{\mathbf{u}}=\frac{\mathbf{u}}{\|\mathbf{u}\|_{2}}=\left[\begin{array}{c}
\sin \beta \cos \gamma \\
\cos \beta \cos \gamma \\
\sin \gamma
\end{array}\right]=\hat{\mathbf{u}}(\beta, \gamma)
$$



Figure 1: Reference frames and control vector decomposition
Lastly, for reasons made explicitly clear in $\S 2.4$, neither the ascending node ( $\Omega$ ) nor the argument of periapsis $(\omega)$ are directly considered in the analysis, leaving the semi-major axis $a$, the eccentricity $e$, and inclination $i$ as the remaining state-space components. Thus, after all of the above assumptions and choices are accounted for the final form of the $\Delta v$ HJB PDE to be obtained

$$
\begin{equation*}
\frac{\partial \tilde{V}}{\partial \Delta v}+\sup _{f, \beta, \gamma}\left[\frac{\partial \tilde{V}^{T}}{\partial \mathbf{\propto}} \mathbf{f}(\propto) \hat{\mathbf{u}}(\beta, \gamma)\right]=0 \tag{7}
\end{equation*}
$$

where here $\propto^{T}=\left[\begin{array}{lll}a & e & i\end{array}\right]^{T}$. The optimal control policy $\left(f^{*}, \beta^{*}, \gamma^{*}\right)$ for the $\Delta v$ range computation
problem in terms of the states $a, e$, and $i$ and adjoints $p_{a}, p_{e}$, and $p_{i}$ is

$$
\left(f^{*}, \beta^{*}, \gamma^{*}\right)=\left\{\begin{array}{l}
\frac{(0,0,0) \text { if } p_{a} \geq 0, p_{a} \geq \frac{-a e}{1-e^{2}} p_{e},}{\text { and }-2\left(a p_{a}+(1-e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}} \leq p_{i} \leq 2\left(a p_{a}+(1-e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}}} \\
\frac{(\pi, 0,0) \text { if } p_{a}<0, p_{a} \leq \frac{-a e}{1-e^{2}} p_{e}}{\text { and } 2 \frac{1-e}{1+e}\left(-a p_{a}+(1+e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}} \leq p_{i} \leq-2 \frac{1-e}{1+e}\left(-a p_{a}+(1+e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}}} \\
\frac{(0, \pi, 0) \text { if } p_{a} \geq 0, p_{a}<\frac{-a e}{1-e^{2}} p_{e}}{\text { and } 2\left(a p_{a}+(1-e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}} \leq p_{i} \leq-2\left(a p_{a}+(1-e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}}} \\
\frac{(\pi, \pi, 0) \text { if } p_{a}<0, p_{a}>\frac{-a e}{1-e^{2}} p_{e}}{\text { and }-2 \frac{1-e}{1+e}\left(-a p_{a}+(1+e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}} \leq p_{i} \leq 2 \frac{1-e}{1+e}\left(-a p_{a}+(1+e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}}} \\
\frac{\left(\pi, 0,-\frac{\pi}{2}\right)}{p_{i}>-2\left(a p_{a}\right.} \text { if } p_{i}>2\left(a p_{a}+(1-e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}}, \text { and } p_{i}>2 \frac{1-e}{1+e}\left(-a p_{a}+(1+e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}} \\
\frac{\left(\pi, 0, \frac{\pi}{2}\right) \text { if } p_{i}>-2\left(a p_{a}+(1-e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}}, p_{i}>2 \frac{1-e}{1+e}\left(-a p_{a}+(1+e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}}}{p_{i}>2\left(a p_{a}+(1-e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}}, \text { and } p_{i}>-2 \frac{1-e}{1+e}\left(-a p_{a}+(1+e) p_{e}\right)\left(p_{a}^{2}+p_{e}^{2}\right)^{\frac{1}{2}}} \tag{8}
\end{array}\right.
$$

Table 1 summarizes each of the 6 optimal maneuvers indicated by the optimal control policy. Also, after

Table 1: Optimal Control Policy Equivalence

| Case | $f^{*}, \beta^{*}, \gamma^{*}$ | Description |
| :---: | :---: | :--- |
| $H_{1}$ | $0,0,0$ | Raise apoapsis |
| $H_{2}$ | $\pi, 0,0$ | Raise periapsis |
| $H_{3}$ | $0, \pi, 0$ | Lower apoapsis |
| $H_{4}$ | $\pi, \pi, 0$ | Lower periapsis |
| $H_{5}$ | $\pi, 0,-\frac{\pi}{2}$ | Increase $i$ at apoapsis |
| $H_{6}$ | $\pi, 0, \frac{\pi}{2}$ | Decrease $i$ at apoapsis |

significant simplification the resulting state dynamics are found

$$
\begin{gather*}
\frac{\mathrm{d} a}{\mathrm{~d} \Delta v}=\frac{a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}\left(1-e^{2}\right)^{\frac{1}{2}}}\left[a\left(1+e \cos f^{*}\right)\right] \cos \beta^{*} \cos \gamma^{*}  \tag{9}\\
\frac{\mathrm{~d} e}{\mathrm{~d} \Delta v}=\frac{a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}\left(1-e^{2}\right)^{\frac{1}{2}}}\left[\frac{2\left(1-e^{2}\right)\left(e+\cos f^{*}\right)}{1+e \cos f^{*}}\right] \cos \beta^{*} \cos \gamma^{*}  \tag{10}\\
\frac{\mathrm{~d} i}{\mathrm{~d} \Delta v}=\frac{a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}\left(1-e^{2}\right)^{\frac{1}{2}}}\left[\frac{\left(1-e^{2}\right) \cos \left(\omega^{*}+f^{*}\right)}{1+e \cos f^{*}}\right] \sin \gamma^{*} \tag{11}
\end{gather*}
$$

Along optimal trajectories, the necessary condition for optimality on the adjoint variables $p_{a}, p_{e}$, and $p_{i}$ associated with the states $a, e$, and $i$, is that $\mathrm{d} \mathbf{p} / \mathrm{d} \Delta v=-\partial \mathcal{H} / \partial \mathbf{x}$, yielding

$$
\begin{array}{r}
\frac{\mathrm{d} p_{a}}{\mathrm{~d} \Delta v}=\frac{a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}\left(1-e^{2}\right)^{\frac{1}{2}}}\left[-3\left(1+e \cos f^{*}\right) \cos \beta^{*} \cos \gamma^{*} p_{a}-\frac{\left(1-e^{2}\right)\left(e+\cos f^{*}\right)}{a\left(1+e \cos f^{*}\right)} \cos \beta^{*} \cos \gamma^{*} p_{e}\right. \\
\left.-\frac{1-e^{2}}{a\left(e+\cos f^{*}\right)} \cos \left(\omega^{*}+f^{*}\right) \cos f^{*} \sin \gamma^{*} p_{i}\right] \tag{12}
\end{array}
$$

$$
\begin{gather*}
\left.\frac{\mathrm{d} p_{e}}{\mathrm{~d} \Delta v}=\frac{a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}\left(1-e^{2}\right)^{\frac{1}{2}}\left[-\frac{2 a\left(e+\cos f^{*}\right)}{\left(1-e^{2}\right)} \cos \beta^{*} \cos \gamma^{*} p_{a}+\frac{2 e\left(e+\cos f^{*}\right)}{\left(1+e \cos f^{*}\right)} \cos \beta^{*} \cos \gamma^{*} p_{e}\right.} \begin{array}{rl} 
& +\frac{1+e \cos f^{*}}{\cos f^{*}\left(e+\cos f^{*}\right)^{2}} \cos \left(\omega^{*}+f^{*}\right) \sin \gamma^{*} p_{i}
\end{array}\right] \\
\frac{\mathrm{d} p_{i}}{\mathrm{~d} \Delta v}=0 \tag{13}
\end{gather*}
$$

Taken together, (12), (13), and (14) define the dynamics of the adjoint variables $p_{a}, p_{e}$, and $p_{i}$ in the $\Delta v$ integration space along time-independent, $\Delta v$-optimal trajectories. Interestingly, the optimal control policy (8) is piecewise defined, and regions of optimality in the adjoint space can be used to view the optimal policy using a graph topology, such as shown in Fig. 2.


Figure 2: Optimal basis maneuvers visualized as nodes on a bi-directional graph. Classical maneuver combinations are emphasized.

Interestingly, the optimal control policy (8) found using a first-principles optimal controls approach can exactly reproduce several classical orbital maneuvers, namely the Hohman, Bi-Elliptic, and Bi-Elliptic with plane change transfers. The following section discusses the applicability of the approach described here to $J_{2}$-perturbed dynamics.

### 2.4 Applicability to Orbit Range with $J_{2}$ Perturbations

The the analysis thus far derived the required equations to compute optimal orbit range sets in $a$, $e$, and $i$ using the classical GVE equations of motion. If $J_{2}$ perturbations are also considered, the time dynamics of the mean semi-major axis $\bar{a}$, mean eccentricity $\bar{e}$, and mean inclination $\bar{i}$ have the same form as the unperturbed orbit elements $a, e$, and $i$, while in general (with the exception of particular critical regions) the mean ascending node $\bar{\Omega}$ and mean argument of periapsis $\bar{\omega}$ drift at different rates over time. This property, that $(\bar{a}, \bar{e}, \bar{i})$ still behave as constants of motion while the orientation coordinates $\bar{\Omega}$ and $\bar{\omega}$ drift over time, can be exploited to realize significant $\Delta v$ savings in the general Two-Point Boundary Value Problem (TPBVP). Using these properties, trajectories in regions where $J_{2}$-induced drift rates in $\bar{\Omega}$ and $\bar{\omega}$ are non-zero (and different from one another) can place themselves in intermediate orbits and allow $J_{2}$ perturbations to perturb them until a desired instantaneous value of $\Omega$ and $\omega$ is reached. Because in [A1] it is assumed that the user is unconcerned with the total time of the orbit transfer, this effectively allows the user to transfer from any
initial instantaneous orbit element set $\propto_{0}$ to any final instantaneous orbit element set $\propto_{f}$ incurring only the $\Delta v$ cost associated with transitioning from $\left(a_{0}, e_{0}, i_{0}\right)$ to ( $a_{f}, e_{f}, i_{f}$ ). Thus, the orbit element range set computation method developed in this paper provides the general minimum-fuel, free-time optimal transfer set from an initial orbit to any final orbit under $J_{2}$ perturbations.

## 3 APPLICATIONS

Servicing vehicle placement and optimal servicing tour applications are now discussed. Fig. 3 plots range sets for three initial orbits with $\Delta v=1000 \mathrm{~m} / \mathrm{s}$ to aid the reader in range set visualization. This particular plot corresponds with example 2 in [7].


Figure 3: Range set for three initial orbits (LEO, GTO, and GEO) with a final $\Delta v$ constraint of $1,000 \mathrm{~m} / \mathrm{s}$. Projections of the range set in the $a-e, a-i$, and $e-i$ planes are also shown along with markers indicating initial orbits to aid in visualization.

Note that the Low Earth Orbit (LEO) range set starting at $a=6700 \mathrm{~km}, e=0$ is not very large. This is entirely expected, as gravitational dynamics are much stronger at smaller semi-major axes. Conversely, both the Geostationary Transfer Orbit (GTO) and Geostationary Orbit (GEO) range sets are much larger. Also, the relative ease of inclination change at higher radii is particularly emphasized in the GTO and GEO range sets.

### 3.1 Servicing Vehicle Placement

An intuitive application of $\Delta v$ range sets are service vehicle placement exercises. Whether client spacecraft are known a-priori or planners desire to place a service vehicle such that the greatest number of potential clients are reachable, $\Delta v$ range sets may be used to visually evaluate candidate placement orbits. A notional illustration of this approach is shown in Fig. 4.


Figure 4: Notional illustration of how $\Delta v$ range set computation can be used to aid service vehicle placement mission planning activities.

In the notional scenario shown in Fig. 4 there are several client orbits of interest $\left(œ_{1}, œ_{2}, \ldots, œ_{n}\right)$. There are three candidate service vehicle placement orbits considered: $\propto_{S / C}^{A}, \propto_{S / C}^{B}$, and $\propto_{S / C}^{C}$. With an arbitrary $\Delta v$ constraint, there are three corresponding range sets (Range set ' A ', ' B ', and ' C '). Upon inspection it is clear that Range Set ' C ' only includes $œ_{1}$ and Range Set ' B ' only includes $\wp_{2}, \ldots, \propto_{n}$. However, Range set ' $A$ ' envelopes all client orbits of interest. This is perhaps the most intuitive and easily accessible application of $\Delta v$ range sets; a casual inspection of range sets and potential clients during mission planning activities requires no special knowledge of optimal controls or even astrodynamics. The complex, multi-dimensional optimal control problem involved in computing fuel-optimal trajectories from an initial orbit to any possible final orbit is neatly visualized as a $\Delta v$ range surface. Objects within or on the edge of the volume defined by these sets are reachable, while those outside are not.

### 3.2 Servicing Tour Optimization

It has been long realized that choosing a fuel-optimal sequence of client spacecraft to service is an instantiation of the Traveling Salesman Problem (TSP) [12, 13]. Because $\Delta v$ is a metric (see $\S 2.1$ ), computing timeindependent, $\Delta v$-optimal tours is a more specific subproblem called the metric-TSP [9]. Specific algorithms have been designed for the Euclidean metric that can identify nearly optimal sequences with a cost factor of $1+1 / c, c>0, c$ arbitrarily large [10]. Using these algorithms, Euclidean distance TSPs have been solved for in excess of 85,000 nodes, far exceeding the 1,300 active spacecraft on orbit [11].

Fig. 5 demonstrates how evaluating tour optimization problems using time-independent, $\Delta v$-optimal trajectories is notionaly accomplished. On the left-hand side, the various client orbits are plotted along with their optimal transfer paths in orbit-element space. Note that in orbit element space, differences in orbits do not satisfy the triangle inequality. On the right-hand side, the client orbits (nodes) and associated transfer differences are transformed onto the $\Delta v$ metric space, where the triangle-inequality does apply. It is on this space that the metric-TSP is solved.

It must be emphasized that computing the $\Delta v$ range set for each of the $n$ nodes requires approximately $\mathcal{O}\left(n k^{3}\right)$ operations, whereas computing all inter-orbit optimal transfers requires only $\mathcal{O}\left(n^{2}\right)$. Unless $n$ is


Figure 5: The optimal connecting trajectories in orbit element space (left) may be represented using a nondirectional graph (right) where the edges between nodes (orbits) are weighted according to $\Delta v$ metric costs associated with such a transfer.
arbitrarily high, to compute the individual distances between client spacecraft (nodes), it is likely much more computationally optimal to simply utilize classical orbit transfer results. Still, constructing the probulem in the $\Delta v$ metric space provides insight into the problem.

## 4 Conclusions \& Future Work

The application of $\Delta v$ range set computation to service vehicle placement and optimal tour design is described and motivated. Existing literature is cited to demonstrate that $\Delta v$ costs from minimum $\Delta v$ connecting trajectories are metrics. Combined with the Generalized Independent Parameter (GIP) Hamilton-JacobiBellman (HJB) PDE, when $\Delta v$ is the independent paramter, resulting solutions are instances of Metric Range Sets. The time-independent, $\Delta v$ optimal control policy is introduced and shown to reproduce classical orbit transfer maneuvers, including the Hohman, Bi-Elliptic, and Bi-Elliptic with plane change transfers. It is argued that the inclusion of $J_{2}$ perturbations allows the range set computation to ignore explicit inclusion of the argument of periapsis $(\omega)$ and ascending node $(\Omega)$, as they secularly drift with time and may be controlled through maneuver timing. Finally, both service vehicle placement and $\Delta v$-optimal tour selection are both introduced as applications for $\Delta v$-optimal range sets.

## References

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