Covariance Based Pre-Filters and Screening Criteria for Conjunction Analysis

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Abstract

Several relationships are developed relating object size, initial covariance and range at closest approach to probability of collision. These relationships address the following questions:

1. Given the objects' initial covariance and combined hard body size, what is the maximum value P_{max} of the collision probability possible?

2. Given the objects' initial covariance, what is the maximum combined hard body radius $r_{A,max}$ for which the collision probability does not exceed the tolerance limit P_{toler}?

3. Given the objects' initial covariance and the combined hard body radius $r_{A,max}$, what is the maximum miss distance $x_{e,max}$ for which the collision probability equals or exceeds the tolerance limit P_{toler}?

The first relationship above allows the elimination of object pairs from conjunction analysis (CA) on the basis of the initial covariance and hard-body sizes of the objects. The application of this pre-filter with a P_c limit of 1×10^{-6} to present day General Perturbations (GP) catalogs with estimated GP covariance results in the elimination of approximately 66% of object pairs as unable to ever conjunct with a probability of collision exceeding the limit. This pre-filter is expected to have a significantly larger impact on future catalogs, which are likely to contain a larger fraction of small debris tracked only by a limited subset of available sensors.

This relationship also provides a mathematically rigorous basis for eliminating objects from analysis entirely based on element set age or quality – a practice commonly based on rough rules of thumb today.

Further, these relations can be used to determine the required geometric screening radius for all objects. This analysis reveals that the screening volumes for small objects are much larger than needed, while the screening volumes for pairs of large objects may be inadequate.

These relationships may also form the basis of an important metric for catalog maintenance by defining the maximum allowable covariance size for effective conjunction analysis.

The application of these techniques promises to greatly improve the efficiency and completeness of conjunction analysis.

1. Introduction

As the satellite catalog grows due to launches, breakups and the advent of new sensor systems capable of tracking smaller objects, the challenge of providing effective and timely conjunction analysis (CA) becomes more computationally difficult. At the same time, what exactly constitutes *effective* CA remains somewhat elusive. This analysis develops covariance based pre-filtering techniques to reduce the computational burden of satellite conjunction screening analysis. Unlike orbital geometry based filters, such as apogee/perigee or orbit path filters, these filters do not eliminate object pairs on the basis of their inability to physically conjunct. Rather, these filters eliminate objects on the basis that the available position covariance information is inadequate to perform an effective risk assessment, as represented by probability of collision (P_c).

As such, the effectiveness of these metrics is inversely proportional to the accuracy of the catalog and may serve to help define accuracy requirements for the space catalog. Additional aspects of the analysis will help to define requirements for conjunction screening parameters such as maximum element set age and conjunction screening radius. Three relationships are presented and applied to a public General Perturbations (GP) catalog of 14,642 objects from August 2012.

1.1 Notation

C_p Covariance matrix of Primary object expressed in its own principal axes coordinate system

$$C_p = \begin{bmatrix} \sigma_{x,p}^{\ 2} & 0 & 0 \\ 0 & \sigma_{y,p}^{\ 2} & 0 \\ 0 & 0 & \sigma_{z,p}^{\ 2} \end{bmatrix} \quad \text{where, by definition,} \quad \sigma_{x,p} > \sigma_{y,p} > \sigma_{z,p}\,. \tag{1}$$

C_s Covariance matrix of Secondary object expressed in its own principal axes coordinate system

$$C_s = \begin{bmatrix} \sigma_{x,s}^{\ 2} & 0 & 0 \\ 0 & \sigma_{y,s}^{\ 2} & 0 \\ 0 & 0 & \sigma_{z,s}^{\ 2} \end{bmatrix} \quad \text{where, by definition,} \quad \sigma_{x,s} > \sigma_{y,s} > \sigma_{z,s}\,. \tag{2}$$

r_p Radius of Primary Object

r_s Radius of Secondary Object

1.2 Basic Equations

The encounter plane, (x', z') is defined as the plane perpendicular to the encounter relative velocity vector. Let r_A and x_e respectively be the combined radius and the nominal miss distance. Let the secondary be at the origin and the primary be at the nominal position (x'_p, z'_p) as shown in Figure 1.

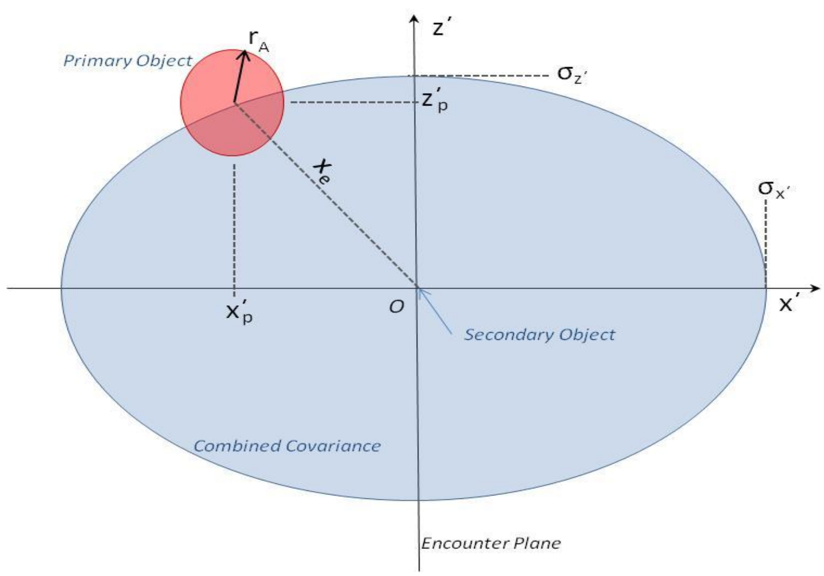

Figure 1 –Encounter Plane

The following are Equations (5.22) through (5.26) from *Spacecraft Collision Probability*[1]. The collision probability P, to first order approximation, is given by

$$P = e^{-v/2}(1 - e^{-u/2}) \tag{3}$$

$$r_A = r_p + r_s \tag{4}$$

$$u \equiv \left(\frac{r_A}{\sigma}\right)^2 \tag{5}$$

$$v \equiv \left(\frac{x_e}{\sigma *}\right)^2 \tag{6}$$

$$\sigma^2 \equiv \sigma_{x'}\sigma_{z'} \qquad \text{and} \qquad \sigma *^2 \equiv \sigma_{z'}{}^2 \left\{1 + \left[\left(\frac{\sigma_{z'}}{\sigma_{x'}}\right)^2 - 1\right]\left(\frac{x'^2_p}{x'^2_p + z'^2_p}\right)\right\}^{-1} . \tag{7}$$

It is convenient to write Equation (7) devoid of the non-essential singularity at $x'_p = 0$ and $z'_p = 0$ as simply

$$v = \frac{x'^2_p}{\sigma_{x'}{}^2} + \frac{z'^2_p}{\sigma_{z'}{}^2} . \tag{8}$$

2 Analysis

2.1 Maximum Probability

Given the object sizes and covariances, what is the maximum value P_{max} of the collision probability possible?

We note that the maximum of Equation (3) is given by placing the primary object at the origin of the encounter plane ($x_e = 0$) thus reducing Equations (3) and (7) to the following:

$$\begin{aligned} P &= \left(1 - e^{u/2}\right) \\ u &= \left(\frac{R_A}{\sigma}\right)^2 \\ \sigma^2 &= \sigma_{x'}\sigma_{z'} \end{aligned} \tag{9}$$

In general, the principal axes of the primary do not align with those of the secondary. We choose an orientation which maximizes the collision probability if we obtain the combined covariance by aligning the two individual covariances. This yields the most peaked combined probability *density* function (pdf) at the origin. Thus, by using Equations (1) and (2), we set

$$\begin{aligned} \sigma_x^2 &= \sigma_{x,p}^2 + \sigma_{x,s}^2 \\ \sigma_y^2 &= \sigma_{y,p}^2 + \sigma_{y,s}^2 \\ \sigma_z^2 &= \sigma_{z,p}^2 + \sigma_{z,s}^2 \end{aligned} \tag{10}$$

Hence, by definition, we have

$$\sigma_x > \sigma_y > \sigma_z \ . \tag{11}$$

Therefore, the (x′, z′)-encounter plane as shown in Figure 1 is now the (y,z)-encounter plane shown in Figure 2. For the typical satellite covariance orientation (major axis aligned with the velocity vector) this corresponds to head-on or overtaking conjunctions.

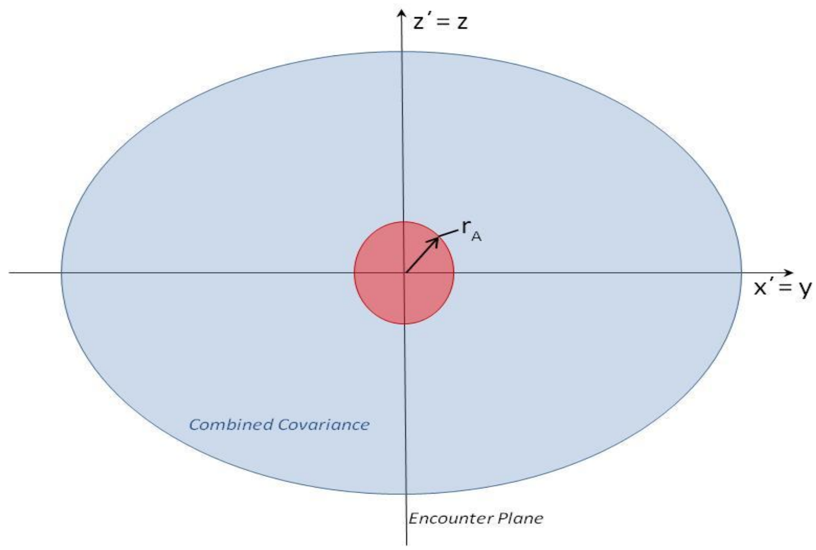

Figure 2 – Maximum Probability Encounter Plane

Because we are specifically considering the center of the encounter plane, the maximum collision probability P_{max} decreases when the covariance size increases. Assuming that covariance size increases monotonically with time, the P_{max} calculated at the beginning of an analysis window is the maximum value for the entire analysis time frame. Therefore, if P_{max} for a pair of objects is less than the probability of collision screening limit (P_{toler}) at the beginning of the analysis, this pair of objects can be disregarded for the duration of the analysis.

Using this information, we compare P_{max} to the tolerance limit P_{toler}. If $P_{max} < P_{toler}$, then we eliminate the conjunction from further consideration.

2.2 Maximum Hard Body Radius

Given the covariance size, what is the maximum hard body radius $r_{A,max}$ for which the collision probability does not exceed the tolerance limit P_{toler}?

We note that the encounter plane for this case is also given by Figure 2, except that we solve the first member of Equation (9) for $r_{A,max}$ in terms of P_{toler} substituted for P_{max}. Thus, we obtain

$$r_{A,max} = \sigma\sqrt{-2\left[\ln\left(1-P_{toler}\right)\right]} \tag{12}$$

Thus, for a given pair of objects, if $r_A < r_{A,max}$, then the collision probability $P < P_{toler}$ and the pair of objects can be eliminated from further consideration.

2.3 Maximum Miss Distance

Given the covariance and the hard body radius, what is the maximum miss distance $x_{e,max}$ for which the collision probability equals or exceeds the screening limit, P_{toler}?

We note that the upper bound to the miss distance x_e is given by the condition that the primary is nominally on the (major) x'-axis of the encounter plane as shown in Figure 3.

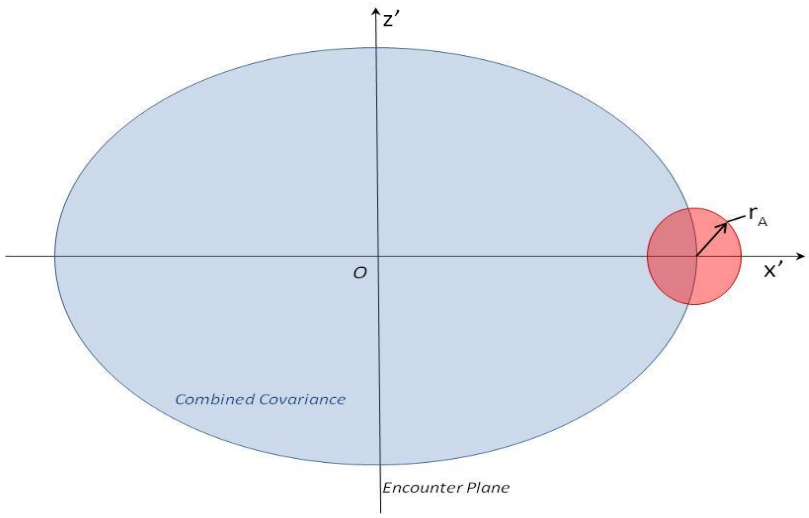

Figure 3 – Maximum Miss Distance Encounter Plane

We compute the maximum range, $x_{e,max}$, for which the collision probability equals or exceeds the tolerance limit by setting $P = P_{toler}$ in Equations (3) through (7). Thus, we have

$$\begin{aligned}
&P_{toler} = e^{-v/2}(1-e^{-u/2}) \\
&u=\left(\frac{r_A}{\sigma}\right)^2 \\
&v=\left(\frac{x_{e,max}}{\sigma^*}\right)^2 \\
&\sigma^2 = \sigma_{x'}\sigma_{z'} \\
&\sigma^* = \sigma_{x'}\,.
\end{aligned} \tag{14}$$

By substituting u, v, σ and σ*into the first member of Equation (14) and then solving for $x_{e,max}$ in terms of r_A and P_{toler} we obtain

$$x_{e,max} = \sigma_{x'}\sqrt{2\left[ln\left(\frac{1-e^{-u/2}}{P_{toler}}\right)\right]} = \sigma_{x'}\sqrt{2[ln(1-e^{-u/2}) - ln(P_{toler})]} \tag{15}$$

Note that the choice of the encounter plane to maximize Equation (15) is now complicated by the presence of $\sigma_{x'}$ in the denominator of u and as the coefficient of the radical. To deal with this, we first simplify the analysis by assuming that the orbits of the primary and the secondary are circular and that for each orbit, the major axis (x_p or x_s) of the covariance ellipsoid is aligned with the velocity vector, the minor axis (z_p or z_s) is in the nadir direction, so that the intermediate axis(y_p or y_s) is in the cross-track direction. Let the primary and secondary coordinate systems be separated by an arbitrary rotation, β, about their common z-axis as shown in Figure 4, where the angle β is measured from the velocity vector of the secondary to that of the primary with a range $0 \leq \beta < 2\pi$.

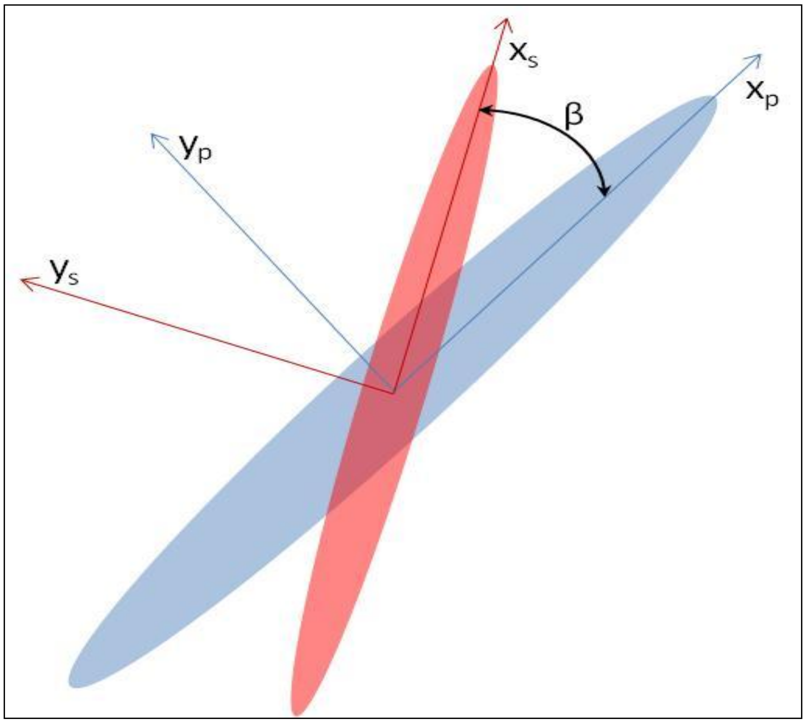

Figure 4 – Relative Orientation of Covariance Ellipsoids

For each value of β, we obtain the combined covariance C (expressed in the local coordinate system of the primary) according to Equations (2.1) and (2.6) of *Spacecraft CollisionProbability*[1]:

$$C = C_p + TC_sT^T$$

$$T = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

where the covariances C_p and C_s of the primary and secondary are given by Equations (1) and (2) respectively. Since the two orbits are assumed to be circular with essentially the same altitude (for otherwise there would be no close conjunction), it follows that the two velocities have essentially the same magnitude. A little consideration of Figure 4 reveals that the relative velocity makes an angle of $(\pi - \beta)/2$ with the x_p-axis. Because the encounter (x',z')-plane is perpendicular to the relative velocity, therefore the encounter plane makes an angle of $\beta/2$ with the x_p-axis and passes through the origin of the combined covariance. Using the same reasoning, we conclude that for a relative velocity perpendicular to the first one just considered, the corresponding encounter plane is perpendicular to the first encounter plane. It follows that this second encounter plane is actually the (y',z')-plane already obtained for the first case. Thus, instead of covering the entire range $0 \leq \beta < 2\pi$, it is efficient to consider the smaller range $0 \leq \beta \leq \pi/2$ and switch the (x', z') and (y', z') planes as appropriate.

For each choice of β, we determine the miss distance $x_{e,max}$ by using Equation (15) and then switch (x′, z′) to (y', z') as just discussed above. The whole process is performed over the range $0 \le \beta \le \pi/2$ to obtain the maximum value $x_{e,max}$ for a given hard body radius r_A.

These results are shown in Figure 5. For this particular case, the choice of (x',z')-plane yields a greater value of the miss distance $x_{e,max}$ for $0 < (\sigma_{z'} / \sigma_{y'})^2 < 0.68$ and the choice of (y',z')-plane yields a greater value of the miss distance $x_{e,max}$ for $0.68 < (\sigma_{z'} / \sigma_{y'})^2 < 1$.

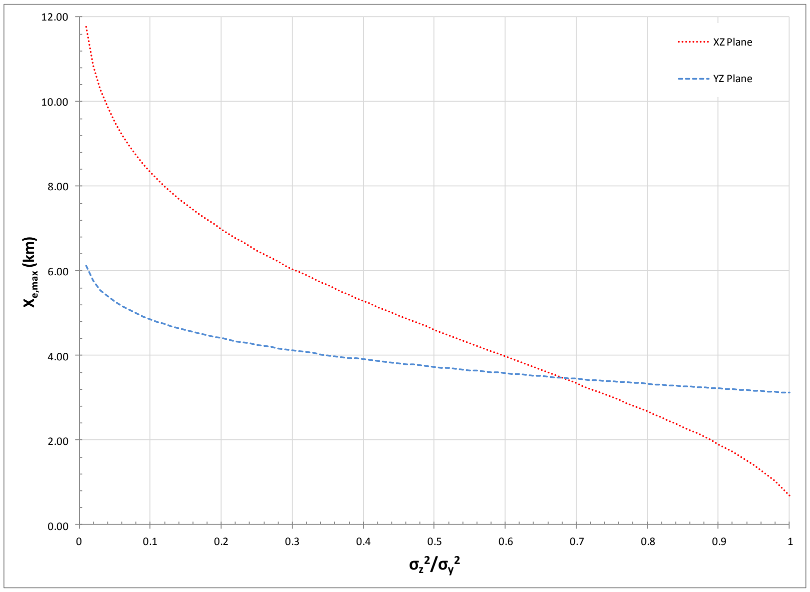

Figure 5 – $x_{e,max}$ as a Function of Covariance Size Ratio

This analysis may now be used to evaluate the required screening radius for any pair of objects.

3 Discussion

The preceding relations are applied to a representative General Perturbations (GP) catalog from Space-Track[2] of 14,642 objects from August 2012. Element set age is between 0 and 15 days at analysis epoch t_0. The GP covariance is estimated using COVGEN[3]. Open source information is used for object sizes when available. Radar Cross Section (RCS) data is used to estimate size for all other objects.

3.1 Maximum Probability

P_{max} was calculated using Equation (9) for every object pair in the study catalog at time t_o (Figure 6). Note that only 34.1% of all object pairs can possibly meet or exceed a P_c limit of 1×10^{-6} at t_0. The remaining 65.9% of all object pairs can be discarded. By comparison, an apogee-perigee filter with a 30 km pad applied to the same catalog results in the elimination of 58% of object pairs. However, note that while the apogee-perigee filter determines that an object pair cannot collide, a P_{max} filter only determines that the available covariance data is not sufficient to determine if the risk of collision exceeds the threshold P_{toler}.

This analysis was repeated with the estimated GP covariance artificially scaled down by a factor of 2 to indicate the impact of catalog covariance size on this metric. In this case, 49.2% of all object pairs can possibly meet or exceed a P_c limit of 1×10^{-6} at t_0. Next, the analysis was repeated with the size of all objects scaled down by a factor of 2 to illustrate the impact of object size. For this case, 21.8% of all object pairs can possibly equal or exceed a P_c limit of 1×10^{-6} at t_0.

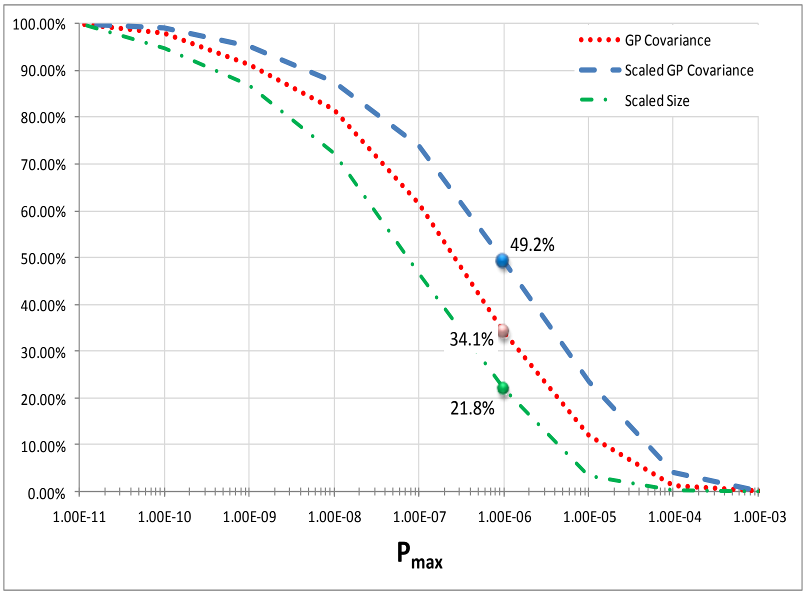

Figure 6 – Fraction of Object Pairs Exceeding P_{max}

The calculation for Figure 6 was now repeated for $(t_0 + \Delta t)$ for a P_c limit of 1×10^{-6} to illustrate the impact of element set age and covariance growth. This data is presented in Figure 7.

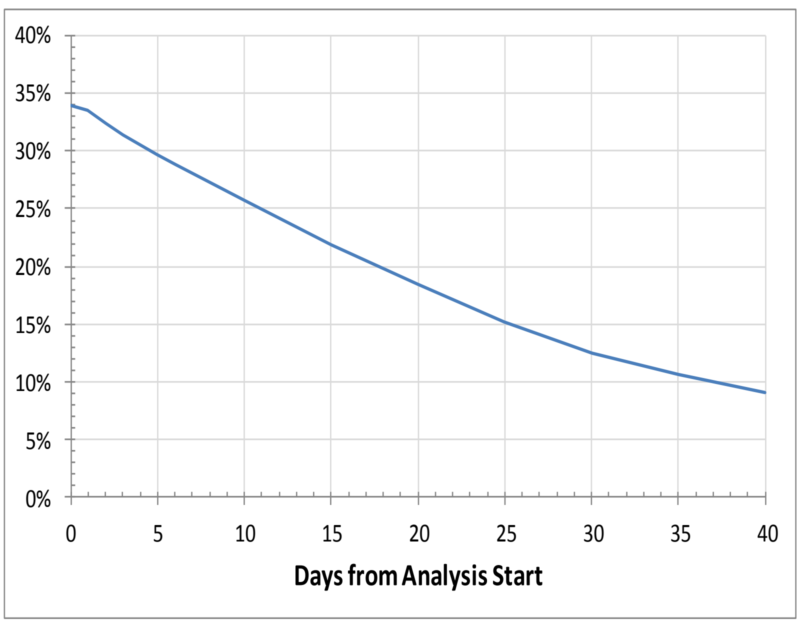

Figure 7 – Fraction of Object Pairs with P Exceeding 1×10^{-6}

Figure 7 suggests that this analysis may be useful for determining when an object's element set becomes too old to be included in Conjunction Analysis (CA) screening.

The P_{max} calculation was compared to 100,000 unique conjunction events (GP data, range $\leq$ 10 km, $P_c \geq 1\times10^{-6}$) identified by the CSieve[4] CA screening tool. The P_{max} results, calculated at both the analysis start time and time of closest approach, were always greater than the P_c calculated for the event. This serves as a check on the derivation and implementation of the P_{max} calculation and confirms the utility of the calculation as a CA screening pre-filter.

3.2 Maximum Range

The maximum range, R_{max}, for a probability limit of 1×10^{-6} was calculated using Equation (15) for every object pair in the study catalog at time t_o (Figure 8). Note the long tail on the graph indicating that a significant fraction of object pairs should be examined at ranges far beyond the values typically used. Conversely, nearly 30% of object pairs cannot exceed $P_c = 1\times10^{-6}$ at a range of 1 km. Again, the analysis is repeated with the covariance scaled down by a factor of 2 to illustrate the impact of covariance size.

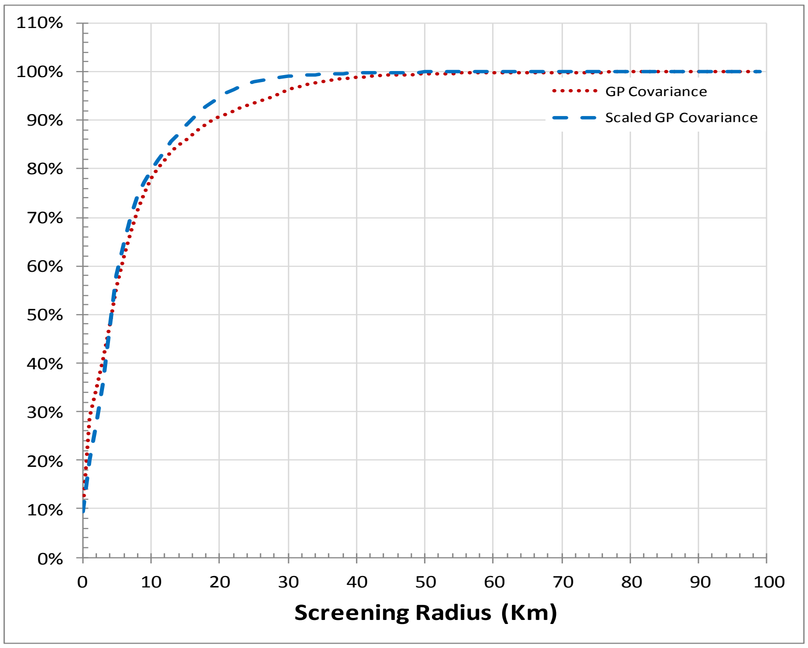

Figure 8 – Fraction of Object Pairs Exceeding $P_c = 1\times10^{-6}$ at a Given Screening Radius

The maximum range calculation was compared to 100,000 unique conjunction events (GP data, range ≤ 10 km, $P_c \geq 1\times10^{-6}$) identified by the CSieve CA screening tool. The maximum range results calculated at at the time of closest approach, were always greater than the event miss distance. However, the maximum range results calculated at the analysis start epoch were less than the event miss distance in 2% of the cases. This discrepancy was found to be due to the change in covariance aspect ratios over the time span from the analysis start time to the time of closest approach. This serves as a check on the derivation and implementation of the maximum range calculation and illustrates the increased complexity of applying the maximum range calculation compared to the P_{max} calculation.

3.3 Impact of Outliers

Despite only 34% of object pairs being capable of reaching $Pc \geq 1x10^{-6}$, the P_{max} calculation does not eliminate any objects from the analysis completely. Similarly, the max range analysis produces very large required screening volumes. Both of these phenomena are caused by a handful of large objects with very large in-track covariance - typically rocket bodies or large payloads in highly eccentric orbits. Options to handle this phenomena include screening these outliers separately and focused efforts to improving covariance for these objects.

4 Conclusion

The analysis presented provides significant insight and opportunity for the All vs. All conjunction analysis (CA) problem. The maximum probability analysis describes a new, powerful and efficient CA pre-filter. This analysis is very effective at screening events from further processing, with only 34% of the object pairs in the test catalog passing the filter. However this effectiveness highlights the critical impact of catalog quality on the effectiveness of conjunction analysis. This filter does not indicate that only 34% of the object pairs are at risk of collision, rather it indicates that 34% of the object pairs have accurate enough data to be effectively analyzed by a typical risk metric (P_c). This clearly illustrates the ineffectiveness of GP quality data for conjunction analysis.

It is expected that the space catalog may grow to on the order of 100,000 objects, with much of this growth coming from the ability of new sensors to track smaller debris. However, the smaller size of these objects and larger covariance due to the limited sensor coverage suggests that the addition of these objects may not have as significant an impact on CA computational processes as may be expected.

Additionally, this analysis will allow analysts to set rigorous criteria for various input parameters to the CA process – most notably maximum element set age or quality and the range screening threshold. The latter may suggest a new way to partition CA screening - the fact that 30% of all object pairs cannot exceed $P_c = 1\times10^{-6}$ at a range of 1 km suggests that significant improvements in screening performance may be possible by tailoring screening radius on a per-object basis.

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[1] F. Kenneth Chan, *Spacecraft Collision Probability*, (American Institute of Aeronautics and Astronautics, 2008), pp14-15, 82

[2] Space-Track. 2004. United States Air Force, Strategic Command. Aug. 2012 <https://www.space-track.org>.

[3] Peterson, G.; Gist, R.; Oltrogge, D., "*Covariance Generation for Space Objects Using Public Data*", Proceedings of the 11th Annual AAS/AIAA Space Flight Mechanics Meeting, Santa Barbara, CA; United States; 11-15 Feb. 2001. pp 201-214. 2001

[4] George, E., "*A High Performance Conjunction Analysis Technique for Cluster and Multi-Core Computers*", Advanced Maui Optical and Space Surveillance Technologies Conference, 13-16 Sep. 2011, Wailea, Maui, HI