One Class of Nonlinear Model Solutions for Flight Vehicles and Applications to Targeting and Guidance Schemes

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Derivation and integration of the 3-rd order vector differential equation of aircraft motion, and its applications to various dynamical models are presented. It is assumed that (a) acceleration due to aerodynamic lift, and the difference between the propulsive thrust and aerodynamic drag accelerations are not changed; (b) the bank angle is zero; (c) the sideslip angle is zero. The general integral and the corresponding analytical solutions for a class of non-steady flight trajectories consist of six independent integrals for heading angle, magnitude of velocity vector, time, altitude, and two components of the position vector. Explicitness with respect to the problem parameters can make these expressions useful in the design of trajectories, and the targeting and guidance schemes. Similarity in the corresponding models makes the first integrals valid for aircraft, spacecraft, re-entry vehicles and missiles. An illustrative example has shown that the general integral provides a wide range of trajectories with various terminal conditions.

Nomenclature

$C_L,\ C_D$ - aerodynamic lift and drag coefficients

$\mathbf{D},\ \mathbf{L}$ - aerodynamic drag and lift

$\mathbf{T}_F^E$ - transformation matrix from F-frame to E-frame

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$\mathbf{T}$ - thrust

$\mathbf{W}$ - weight

$\mathbf{a}_T,\ \mathbf{a}_D,\ \mathbf{a}_L$ - propulsive thrust, aerodynamic drag and lift accelerations

H_1 - difference between thrust component and aerodynamic drag accelerations, $[m/s^2]$

H_2 - acceleration due to thrust component and aerodynamic lift, $[m/s^2]$

$\mathbf{e}, \mathbf{e}_i^F\ (i = 1, 2, 3)$ - unit vectors of F-frame

$\mathbf{j}$ - jerk vector

h - vertical coordinate or altitude, $[m]$

$\mathbf{g},\ g_0$ - gravitational acceleration vector and its magnitude, $[m/s^2]$

m - mass, $[kg]$

$c_i,\ i = 1, ..., 6$ - integration constants

$\mathbf{r}^E$ - position vector in Earth centered inertial frame

t - time, $[s]$

v - magnitude of velocity vector, $[m/s]$

$\beta,\ \phi,\ \psi$ - sideslip, bank and heading (velocity yaw) angles respectively, $[rad]$

β_m - ballistic coefficient

$\xi,\ \eta$ - horizontal coordinates, $[m]$

γ - flight path angle, $[rad]$

$\boldsymbol{\omega}$ - angular velocity vector

Introduction

The studies presented in this paper are devoted to derivation and integration of the 3-rd order differential equation [1],

$$\frac{d\ddot{\mathbf{r}}}{dt} = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^2} \times (\ddot{\mathbf{r}} - \mathbf{g}), \tag{1}$$

obtained for the nonlinear model of the aircraft tracking problem under the following assumptions: (1) acceleration due to aerodynamic lift, and the difference between the accelerations due to propulsive thrust and aerodynamic drag are not changed; (2) the aircraft body rate about the roll axis is zero; (3) the angle of attack and the sideslip angle are zero [1]. This equation represents a kinematics equation and plays a central role in the development of the aircraft maneuver model, the aircraft state estimation and prediction schemes for geometric nonlinear filter design and in the analysis of a re-entry vehicle's dynamical model. It is also important to note that Eq.(1) allows us to represent the jerk vector, and its expression does not explicitly depend on the accelerations due to thrust, drag and lift. Analysis show, however, that as the drag is a function of the square of the velocity, it would be very difficult to hold the lift, and the thrust-drag accelerations constant with zero angle of attack. In this paper, it will be shown that Eq.(1) can be derived using the Newton's second law and completely integrated in a closed-form for the following more general assumptions: (a) accelerations due to the thrust component minus drag and the other thrust component plus aerodynamic lift are constant and non-zero; (b) the bank angle is not changed; (c) the sideslip angle is zero. This means that Eq.(1) is also valid for a non-zero and variable angle of attack. It is demonstrated that the assumptions (a-b) can significantly extend the applicability of Eq.(1).

Equation of Nonlinear Maneuver Model

Consider the F-frame formed by the triad of orthogonal unit vectors $\mathbf{e}_1^F$, $\mathbf{e}_2^F$, $\mathbf{e}_3^F$ and with the origin at the aircraft COG: the unit vector $\mathbf{e}_1^F$ is aligned with the velocity vector, the unit vector $\mathbf{e}_3^F$ is aligned long the lift vector and $\mathbf{e}_2^F$ completes the right handed system (see Fig.1). Let us also introduce an inertial ground axes system, $E\xi\eta\zeta$, which will be referred as E-frame. In this paper, it is assumed that the $\mathbf{e}_1^F\mathbf{e}_3^F$- plane is the vertical plane that contains the non-steady flight trajectory, and it forms an angle, ψ with the ξ ζ -plane of the $E\xi\eta\zeta$ coordinate system. In flight mechanics, this angle is called as a heading angle and determined according to the equation [2]:

$$\dot{\psi} = \frac{g_0}{Wv\cos\gamma}(T\sin\bar{\alpha} + L)\sin\phi, \tag{2}$$

where $\bar{\alpha} = \alpha + \alpha_T$ is the angle between the thrust vector and the velocity vector. It will be assumed below that the change of mass of the vehicle is negligible and therefore, the equation for mass, W will not be considered in this paper. If ψ is constant, then from Eqs.(2) one can obtain that $\phi = 0$ assuming $T\sin\bar{\alpha} + L \neq 0$. Note that these conditions are based on the assumptions (a) and (b). If $\boldsymbol{\omega}^F$ is defined as the aircraft angular rate vector and $\omega_1 = \dot{\phi}$ is its component on $\mathbf{e}_1^F$, then $\dot{\phi} = \mathbf{e}_1^F\boldsymbol{\omega}^F$ and $\dot{\mathbf{e}}_1^F = \boldsymbol{\omega}^F \times \mathbf{e}_1^F$ (see Fig. 1). By forming the cross product $\mathbf{e}_1^F \times \dot{\mathbf{e}}_1^F$, one can also find that

$$\boldsymbol{\omega}^F = \dot{\phi}\mathbf{e}_1^F + \mathbf{e}_1^F \times \dot{\mathbf{e}}_1^F. \tag{3}$$

From the definition of $\mathbf{e}_1^F$ it follows that

$$\mathbf{e}_1^F = \frac{\dot{\mathbf{r}}^E}{|\dot{\mathbf{r}}^E|}, \qquad \dot{\mathbf{e}}_1^F = \frac{\ddot{\mathbf{r}}^E}{|\dot{\mathbf{r}}^E|} - \frac{\dot{\mathbf{r}}^E\ddot{\mathbf{r}}^E}{|\dot{\mathbf{r}}^E|^3}\dot{\mathbf{r}}^E. \tag{4}$$

where $\mathbf{r}^E$ is the radius vector of the aircraft COG. The superscript and the subscript E will be omitted below for simplicity. If it is assumed that there is no rotation about $O\mathbf{e}_1^F$ axes, that is $\dot{\phi} = 0$ (assumption b), then substitution of Eqs.(4) into Eq.(3) yields

$$\boldsymbol{\omega}^F = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^2}. \tag{5}$$

Here, the expression of the Newton's second law with transformation from the E-frame to the F-frame and its subsequent differentiation yield:

$$\mathbf{T}_F^E \dot{\mathbf{T}}_E^F (\ddot{\mathbf{r}} - \mathbf{g}) + \mathbf{T}_F^E \mathbf{T}_E^F (\dddot{\mathbf{r}} - \dot{\mathbf{g}}) = \mathbf{T}_F^E \dot{\mathbf{a}}^F, \tag{6}$$

where by the assumptions (a-c),

$$\mathbf{a}^F = H_1 \mathbf{e}_1^F + H_2 \mathbf{e}_3^F, \quad \dot{\mathbf{a}}^F = (\dot{H}_1 - H_2\dot{\gamma})\mathbf{e}_1^F + (\dot{H}_2 + H_1\dot{\gamma})\mathbf{e}_3^F \tag{7}$$

with

$$H_1 = \frac{g_0}{W}(T\cos\bar{\alpha} - D) = const, \qquad H_2 = \frac{g_0}{W}(T\sin\bar{\alpha} + L) = const. \tag{8}$$

Noting that $\mathbf{T}_F^E \dot{\mathbf{T}}_E^F \mathbf{z} = -\boldsymbol{\omega}^F \times \mathbf{z}, \quad \forall \mathbf{z}$, one can rewrite Eq.(6) with respect to the jerk vector:

$$\mathbf{j} = \dddot{\mathbf{r}} = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^2} \times (\ddot{\mathbf{r}} - \mathbf{g}) + \dot{\mathbf{g}} + \dot{\mathbf{a}}^E \tag{9}$$

where $\dot{\mathbf{a}}^E(\dot{a}_1^E, \dot{a}_2^E, \dot{a}_3^E) = \mathbf{T}_F^E \dot{\mathbf{a}}^F$. If $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_h$ are the unit vectors of the E-frame, then it can be shown that (see Fig. 1)

$$\begin{aligned}
\mathbf{e}_1^F &= \cos\psi_0 \cos\gamma \mathbf{i}_\xi + \sin\psi_0 \cos\gamma \mathbf{i}_\eta + \sin\gamma \mathbf{i}_\zeta, \\
\mathbf{e}_2^F &= \sin\psi_0 \mathbf{i}_\xi + \cos\psi_0 \mathbf{i}_\eta, \\
\mathbf{e}_3^F &= -\cos\psi_0 \sin\gamma \mathbf{i}_\xi + \sin\psi_0 \sin\gamma \mathbf{i}_\eta + \cos\gamma \mathbf{i}_\zeta.
\end{aligned}$$

Now assume that the components of $\mathbf{a}^F$, and $\mathbf{a}^E$ and $\mathbf{g}$ in the F-frame and E-frame respectively are not changed during the motion. This yields the following expressions (assumption a):

$$\dot{\mathbf{g}} = 0, \quad \dot{H}_1 = \dot{a}_1^F = 0, \quad \dot{H}_2 = \dot{a}_3^F = 0, \quad \dot{a}_1^E = 0, \quad \dot{a}_2^E = 0, \quad \dot{a}_3^E = 0. \tag{10}$$

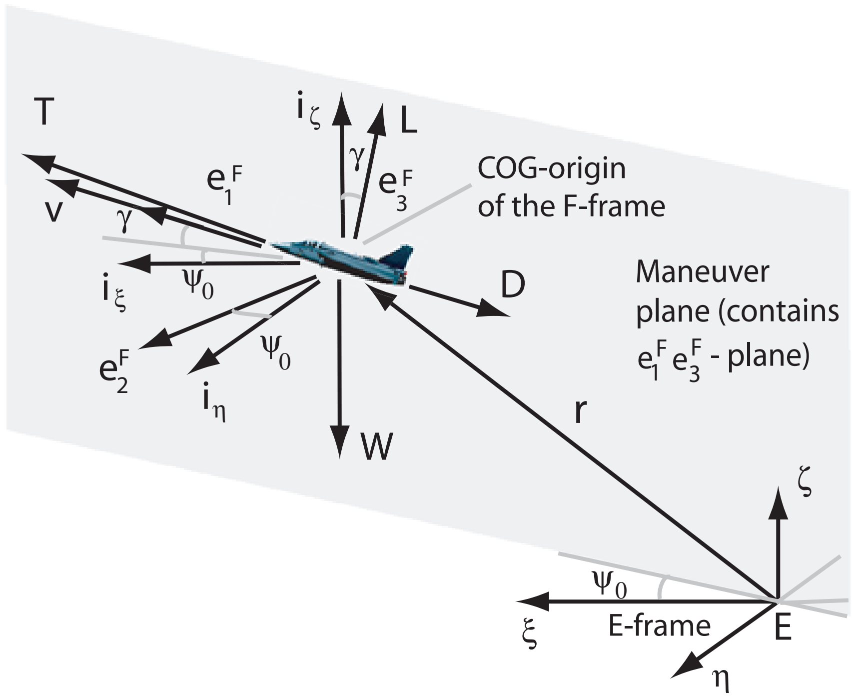

Figure 1: To the nonlinear aircraft model

Note that the conditions for $\dot{a}_1^E$, $\dot{a}_2^E$ and $\dot{a}_3^E$ can be satisfied by appropriate selection of the elements of $\mathbf{T}_F^E$. In this case, Eq.(9) is rewritten as

$$\mathbf{j} = \dddot{\mathbf{r}} = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^2} \times (\ddot{\mathbf{r}} - \mathbf{g}). \tag{11}$$

The importance of this equation is that under the assumptions Eq.(10), the jerk vector does not explicitly depend on the accelerations due to thrust, drag and lift. Eq.(11) is the same as the equation of the aircraft nonlinear model obtained in Ref.[1].

Integrals for Nonlinear Model of Aircraft

Consider the nonlinear maneuver model given by Eq.(1) with the assumptions (a-c). It will be shown below that this equations can be integrated to obtain closed-form solutions in terms of elementary and transcendental functions of the flight path angle. It is convenient to introduce a new vector, $\mathbf{q}$, defined by $\mathbf{q} = \dot{\mathbf{r}}$. Then from Eq.(11) it follows that

$$\ddot{\mathbf{q}} = \frac{(\mathbf{q} \times \dot{\mathbf{q}}) \times \dot{\mathbf{q}}}{|\mathbf{q}|^2} - \frac{(\mathbf{q} \times \dot{\mathbf{q}}) \times \mathbf{g}}{|\mathbf{q}|^2}. \tag{12}$$

By taking the second derivative from $\mathbf{q} = q\mathbf{e}_1^F$ and employing the definition of a double cross product, Eqs.(12) can be reduced to the expression:

$$\ddot{\mathbf{q}} = \ddot{q}\mathbf{e}_1^F + 2\dot{q}\dot{\mathbf{e}}_1^F + q\ddot{\mathbf{e}}_1^F = [(\mathbf{g}\dot{\mathbf{e}}_1^F) - q|\dot{\mathbf{e}}_1^F|^2]\mathbf{e}_1^F + [\dot{q} - (\mathbf{g}\mathbf{e}_1^F)]\dot{\mathbf{e}}_1^F, \tag{13}$$

which yields the following first integral of Eq.(1):

$$\dot{q} = -g_0 \sin(\gamma) + H_1, \tag{14}$$

where H_1 is the integration constant and $\mathbf{g} = -g_0\mathbf{i}_\varsigma$. In the same manner, by taking the first derivative of $\mathbf{q} = q\mathbf{e}_1^F$ and using the Newton's second law, one can obtain the expression

$$\dot{\mathbf{q}} = \frac{d}{dt}(q\mathbf{e}_1) = \dot{q}\mathbf{e}_1^F + q\dot{\gamma}\mathbf{e}_3^F = \ddot{\mathbf{r}} = \mathbf{g} + \mathbf{a}^E,$$

which yields another first integral of Eq.(11):

$$q\dot{\gamma} = -g_0 \cos\gamma + H_2, \tag{15}$$

where H_2 is the integration constant and $\mathbf{g}\mathbf{e}_3^F = -g_0 \cos\gamma$. This means that under the assumptions used above, Eqs.(1) will follow from Eqs.(14) and (15) which represent the integrals for propulsive and aerodynamic accelerations.

Integrals of Equations of Motion

Equations of motion.

Consequently, using the definitions of the coordinate axes $\mathbf{e}_1^F,\ \mathbf{e}_2^F,\ \mathbf{e}_3^F$ and $\mathbf{i}_\xi,\ \mathbf{i}_\eta,\ \mathbf{i}_\zeta$, one can obtain the following system of equations of motion in the vertical plane [2]

$$\begin{aligned}
\dot{\xi} &= v\cos\gamma\cos\psi, \\
\dot{\eta} &= v\cos\gamma\sin\psi, \\
\dot{\zeta} &= v\sin\gamma, \\
\dot{v} &= H_1 - g_0\sin\gamma, \\
\dot{\gamma} &= \frac{1}{v}(H_2 - g_0\cos\gamma), \\
\dot{\psi} &= 0,
\end{aligned} \tag{16}$$

where the last equation provides the first integral with constant c_1:

$$\psi = c_1. \tag{17}$$

Integrals for magnitude of velocity vector.

In this subsection, it will be shown that the equations of Eqs.(16) for v and γ can be explicitly integrated to obtain formulas for v and t in terms of the elementary and transcendental functions of γ. By considering γ as an independent variable instead of time, t, we have $\dot{v} = dv/dt = dv/d\gamma d\gamma/dt$. Then by eliminating $d\gamma/dt$ from Eqs.(16), one can obtain

$$\frac{dv}{d\gamma}v^{-1} = \frac{-g_0\sin\gamma + H_1}{-g_0\cos\gamma + H_2}, \tag{18}$$

which is integrated in the form:

$$v = c_2 \exp\left[\int_{\gamma_0}^{\gamma} \Phi(\gamma) d\gamma\right], \tag{19}$$

where c_2 is the integration constant, and

$$\Phi(\gamma) = \frac{-g_0 \sin\gamma + H_1}{-g_0 \cos\gamma + H_2} = \frac{A + B\cos x}{a + b\sin x} = \Phi(x),$$

with

$$A = H_1, \quad a = H_2, \quad b = -B = -g_0, \quad x = \gamma + \frac{\pi}{2}, \tag{20}$$

Note that in a particular case when $a + b\sin x = 0$, Eqs.(16) describe a motion with constant v and γ. This case is of a very limited theoretical and practical interest, and not considered in this paper. Then the corresponding integral of $\Phi(x)$, after changing the variables, can be reduced to the form [3] (see formula TI(344), 2.552(2), pp.147 of this reference):

$$\int_{x_0-\alpha}^{x-\gamma} \Phi(x)dx = \int_{x_0-v}^{x-\gamma} \frac{A + B\cos x}{a + b\sin x} dx = \frac{B}{b}\ln(a + b\sin x) + A\int_{x_0-\gamma}^{x-\gamma} \frac{dx}{a + b\sin x}, \tag{21}$$

where the last term represents the table integral and has the following forms depending on the values of a and b [3] (see the formula TI(250), 2.555(3), pp.148 of this reference):

$$\begin{aligned} \int_{x_0-\gamma}^{x-\gamma} \frac{dx}{a + b\sin x} &= \frac{2}{d_1}\arctan\frac{a\tan\bar{x} + b}{d_1}, \qquad [a^2 > b^2], \\ \int_{x_0-\alpha}^{x-\gamma} \frac{dx}{a + b\sin x} &= \frac{1}{d_2}\ln\frac{a\tan\bar{x} + d_3}{a\tan\bar{x} + d_4}, \qquad [a^2 < b^2], \\ \int_{x_0-\alpha}^{x-\gamma} \frac{dx}{a + b\sin x} &= \frac{1}{a}\tan\left(\bar{x} - \frac{\pi}{4}\right), \qquad [a^2 = b^2]. \end{aligned} \tag{22}$$

Here $\bar{x} = x/2$ and $a + b\sin x \neq 0$, and the following constants are used:

$$d_1 = \sqrt{a^2 - b^2}, \quad d_2 = \sqrt{b^2 - a^2}, \quad d_3 = b - \sqrt{b^2 - a^2}, \quad d_4 = b + \sqrt{b^2 - a^2}. \tag{23}$$

Substitution of Eqs.(21) and (22) into Eq.(19) yields

$$
\begin{aligned}
v(\gamma) &= c_2(a+b\sin x)^{-1}\exp\left[\frac{2A}{d_1}\arctan\frac{a\tan\bar{x}+b}{d_1}\right], \qquad [a^2>b^2], \\
v(\gamma) &= c_2(a+b\sin x)^{-1}\left[\frac{a\tan\bar{x}+d_3}{a\tan\bar{x}+d_4}\right]^{(A/d_2)}, \qquad [a^2<b^2], \\
v(\gamma) &= c_2(a+b\sin x)^{-1}\ \exp\left[\frac{A}{a}\ \tan\left(\bar{x}-\frac{\pi}{4}\right)\right], \qquad [a^2=b^2].
\end{aligned} \tag{24}
$$

Integrals for time.

Once $v=v(\gamma)$ is determined, the equation of Eqs.(16) for γ can be integrated as:

$$
t=\int_{x_0-\gamma}^{x-\gamma}\frac{v(x)dx}{a+b\sin x}+c_3, \tag{25}
$$

where c_3 is the new integration constant. If (see Eqs.(19) and (24)):

$$
v(x)=c_2\exp\left[A\int_{x_0-\gamma}^{x-\gamma}\frac{dx}{(a+b\sin x)}\right](a+b\sin x)^{-1} \tag{26}
$$

with $A\neq 0$, then by substituting this expression into Eq.(25), the latter can be rewritten as

$$
t=c_2\int_{x_0-\gamma}^{x-\gamma}\exp\left[A\int_{x_0-\gamma}^{x-\gamma}\frac{dx}{a+b\sin x}\right](a+b\sin x)^{-2}dx+c_3. \tag{27}
$$

Below the boundaries of the integrals will be dropped for simplicity. Introducing the new variable y as

$$
\frac{dy}{dx}=\frac{A}{a+b\sin x}, \tag{28}
$$

Eq.(27) is reduced to the form:

$$
t=\frac{c_2}{A^2}\int\exp(y)\frac{dy}{dx}dy+c_3. \tag{29}
$$

By integrating Eq.(29) by parts, one can obtain that

$$\frac{A^2}{c_2}(t-c_3) = \exp(y)\frac{dy}{dx} - \int \exp(y) d\left[\frac{dy}{dx}\right]. \tag{30}$$

The second term on the right side of Eq.(30) can be simplified as follows:

$$\int \exp(y) d\left[\frac{dy}{dx}\right] = -\frac{b\cos x \exp(y)}{(a+b\sin x)} - b^2 \int \frac{\exp(y)dx}{(a+b\sin x)^2} - a\int \frac{b\sin x \exp(y)}{(a+b\sin x)^2}dx. \tag{31}$$

Using Eqs.(27) and (28), the second term on the right side of Eq.(31) is rewritten as

$$b^2 \int \exp(y)\frac{dx}{(a+b\sin x)^2} = \frac{b^2}{c_2}(t-c_3). \tag{32}$$

Using Eqs.(27) and (28) once again (see Eq.(32)), the third term on the right side of Eq.(31) can also be reduced to the form:

$$a\int \exp(y)\frac{b\sin x}{(a+b\sin x)^2}dx = \frac{a}{A}\exp(y) - \frac{a^2}{c_2}(t-c_3). \tag{33}$$

Consequently, substituting Eqs.(32) and (33) into Eq.(31), one can obtain

$$\int \exp(y) d\left[\frac{dy}{dx}\right] = -\exp(y)\frac{b\cos x}{(a+b\sin x)} - \frac{b^2}{c_2}(t-c_3) - \frac{a}{A}\exp(y) + \frac{a^2}{c_2}(t-c_3). \tag{34}$$

Also, substituting the integral in Eq.(34) back into Eq.(30), and by taking into account Eq.(28), one can obtain the formula for t:

$$t = \frac{c_2}{A^2+a^2-b^2}\exp\left[A\int_{x_0-\gamma}^{x-\gamma}\frac{dx}{a+b\sin x}\right]\left(\frac{A+b\cos x}{a+b\sin x} + \frac{a}{A}\right) + c_3, \tag{35}$$

By employing Eq.(22), and depending on a and b, Eq.(35) can be rewritten in the following final forms:

$$\begin{aligned} t &= \frac{c_2}{A^2+a^2-b^2}\exp\left[\frac{2A}{d_1}\arctan\frac{a\tan\bar{x}+b}{d_1}\right]\left(\frac{A+b\cos x}{a+b\sin x}+\frac{a}{A}\right)+c_3, \qquad [a^2>b^2], \\ t &= \frac{c_2}{A^2+a^2-b^2}\exp\left[\frac{A}{d_2}\ln\frac{a\tan\bar{x}+d_3}{a\tan\bar{x}+d_4}\right]\left(\frac{A+b\cos x}{a+b\sin x}+\frac{a}{A}\right)+c_3, \qquad [a^2<b^2], \\ t &= \frac{c_2}{A^2}\exp\left[\frac{A}{a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right]\left(\frac{A+b\cos x}{a(1+\sin x)}+\frac{a}{A}\right)+c_3, \qquad [a^2=b^2], \end{aligned} \tag{36}$$

where $\sin x \neq -1$. As Eq.(11) is a 3rd-order vector differential equation, which describes the motion in the maneuver plane, its complete integration would require to find *six independent first integrals with six scalar integration constants* of motion in the maneuver plane. So far, three independent first integrals and constants have been found above, that is c_1 in Eq.(17), c_2 in Eq.(24) and c_3 in Eq.(36). Note that the constants H_1 and H_2 are not considered as constants of the equations of motion, Eqs.(16).

Integrals for position vector components.

It can be shown that the magnitude of the velocity vector and the flight path angle are not enough to uniquely determine the position of an aircraft in the maneuver plane. To determine the time histories of the aircraft coordinates, it is convenient to consider the first and third equations of Eqs.(16):

$$\dot{\bar{\xi}} = v\cos\theta, \qquad \dot{\zeta} = v\sin\theta, \tag{37}$$

where $\bar{\xi} = \xi/\cos\psi_0$ with $\psi_0 \neq \pi/2+\pi k,\ k=0,1,2,3....$ Once these equations are integrated, the equation for η of Eqs.(16) can easily be integrated to yield $\eta = \bar{\xi}\sin\psi_0 + c_5$, where it is assumed that the constants c_4 and c_6 are allocated for ξ and ζ. Noting that $\gamma = x - \pi/2$, $\dot{\bar{\xi}} = d\bar{\xi}/dx dx/dt$ and $\dot{\zeta} = d\zeta/dx dx/dt$, one can rewrite Eqs.(37) as

$$\frac{d\bar{\xi}}{dx} = v\frac{dt}{dx}\sin x, \qquad \frac{d\zeta}{dx} = -v\frac{dt}{dx}\cos x, \tag{38}$$

Integration of Eqs.(38) yields the aircraft coordinates $\bar{\xi}$ and ζ. To integrate Eqs.(38), first rewrite the formula for v, Eq.(26) in the form:

$$v = c_2 \frac{d\Delta}{dx}, \qquad \Delta = \Delta(x) = \frac{1}{A} \exp\left[A \int \frac{dx}{a + b\sin x} \right]. \tag{39}$$

Using Eq.(39), it can easily be seen that

$$v\dot{x} = c_2 \frac{d\Delta}{dx}\frac{dx}{dt} = c_2 \frac{d\Delta}{dt}.$$

Based on the notations introduced in Eqs.(20), the first two equations of Eqs.(16) can be rewritten in a compact form:

$$\frac{dv(x)}{dt} = A - b\cos x, \qquad c_2 \frac{d\Delta(x)}{dt} = a + b\sin x. \tag{40}$$

Substitution of these expressions into Eqs.(38) yields

$$\begin{aligned} \bar{\xi} - \frac{a}{A}\zeta + \frac{a}{2Ab}v^2 &= \frac{1}{b}\int v^2 dx + p_{10}, \\ -\zeta + \frac{1}{2b}v^2 &= \frac{A}{b}\int v\frac{dt}{dx}dx + p_{20}, \end{aligned} \tag{41}$$

where p_{10} and p_{20} are the integration constants. Below the integrals on the right sides of these equations will be considered. Using Eq.(26) and taking into account $\bar{A} = 2A$, the integral $\int v^2 dx$ can be rewritten as

$$\int v^2 dx = \frac{c_2^2}{(4A^2 + a^2 - b^2)} \exp(z) \left[\frac{2A + b\cos x}{a + b\sin x} + \frac{a}{2A} \right]. \tag{42}$$

Another integral expression, $\int v(dt/dx)dx$ in the second equation of Eq.(41) can also be reduced to a closed form in terms of elementary and transcendental functions. Using Eq.(26) one can obtain:

$$\int v\frac{dt}{dx}dx = \frac{\eta_2^2\exp(z)}{2(A^2+a^2-b^2)}\left[\frac{(A+b\cos x)}{(a+b\sin x)^2} - \frac{1}{2A} + \frac{3a}{(4A^2+a^2-b^2)}\left(\frac{2A+b\cos x}{a+b\sin x} + \frac{a}{2A}\right)\right]. \tag{43}$$

Consequently, the integrals $\int v^2 dx$ and $\int v(dt/dx)dx$ can be expressed in elementary and transcendental functions presented in Eqs.(42) and (43).

Next, one can substitute Eqs.(42) and (43) into Eqs.(41), and find $\bar{\xi}$ and ζ. For simplicity, the following new variables are introduced:

$$D_1(x) = \int v^2 dx, \qquad D_2(x) = \int v\frac{dt}{dx}dx. \tag{44}$$

Then Eqs.(41) yield

$$\begin{aligned} \bar{\xi} &= \frac{1}{b}D_1 - \frac{a}{b}D_2 + \bar{c}_4, \qquad \bar{c}_4 = (p_{10} - \frac{a}{A}p_{20}) \\ \zeta &= -\frac{A}{b}D_2 + \frac{1}{2b}v^2 + \bar{c}_6, \qquad \bar{c}_6 = -p_{20} \end{aligned} \tag{45}$$

and the $\bar{c}_4$ and $\bar{c}_6$ are the new integration constants. Finally, by substituting the expressions for D_1 and D_2 from Eqs.(44), (42) and (43) into Eqs.(45), and using (22), one can reduce Eqs.(45) to the final forms. Consequently, taking into account $\bar{\xi} = \xi/\cos\psi_0$ with $\psi_0 \neq \pi/2 + \pi k,\ k = 0, 1, 2, 3...$, and using the integral $\eta = \bar{\xi}\sin\psi_0 + \eta_5$ of Eqs.(16), one can write the final formulas for the aircraft coordinates subject to the values of a and b:

$$\begin{aligned} \xi(x) &= P\cos c_1 \exp\left[\frac{4A}{d_1}\arctan\frac{a\tan\bar{x}+b}{d_1}\right] + c_4, \qquad [a^2 > b^2], \\ \xi(x) &= P\cos c_1 \exp\left[\frac{2A}{d_2}\ln\frac{a\tan\bar{x}+d_3}{a\tan\bar{x}+d_4}\right] + c_4, \qquad [a^2 < b^2] \\ \xi(x) &= P\cos c_1 \exp\left[\frac{2A}{a}\tan\left(\frac{x}{2} - \frac{\pi}{4}\right)\right] + c_4, \qquad [a^2 = b^2], \end{aligned} \tag{46}$$

and

$$
\begin{aligned}
\eta(x) &= P\sin c_1 \exp\left[\frac{4A}{d_1}\arctan\frac{a\tan\bar{x}+b}{d_1}\right]+c_5, \qquad [a^2>b^2],\\
\eta(x) &= P\sin c_1 \exp\left[\frac{2A}{d_2}\ln\frac{a\tan\bar{x}+d_3}{a\tan\bar{x}+d_4}\right]+c_5, \qquad [a^2<b^2]\\
\eta(x) &= P\sin c_1 \exp\left[\frac{2A}{a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right]+c_5, \qquad [a^2=b^2],
\end{aligned}
\tag{47}
$$

and

$$
\begin{aligned}
\zeta(x) &= Q\exp\left[\frac{4A}{d_1}\arctan\frac{a\tan\bar{x}+b}{d_1}\right]+c_6, \qquad [a^2>b^2],\\
\zeta(x) &= Q\exp\left[\frac{2A}{d_2}\ln\frac{a\tan\bar{x}+d_3}{a\tan\bar{x}+d_4}\right]+c_6, \qquad [a^2<b^2]\\
\zeta(x) &= Q\exp\left[\frac{2A}{a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right]+c_6, \qquad [a^2=b^2]
\end{aligned}
\tag{48}
$$

with

$$
P=K_1\left[\frac{2A+b\cos x}{a+b\sin x}\right]+\left[\frac{K_3+M_2 b\cos x}{(a+b\sin x)^2}\right]+K_2, \quad Q=J_1\left[\frac{2A+b\cos x}{a+b\sin x}\right]+\left[\frac{(J_3+N_2 b\cos x)}{(a+b\sin x)^2}\right]+J_2,
$$

$$
\begin{gathered}
M_1=\frac{\eta_2^2}{b(4A^2+a^2-b^2)}, \quad M_2=\frac{-a\eta_2^2}{2b(A^2+a^2-b^2)}, \quad M_3=\frac{3a}{(4A^2+a^2-b^2)},\\
N_1=\frac{A\eta_2^2}{2b(A^2+a^2-b^2)}, \quad N_2=\frac{3a}{(4A^2+a^2-b^2)}, \quad N_3=n_3\eta_2^2,\\
K_1=M_1+M_2M_3, \quad K_2=\frac{a}{2A}(M_1+M_2M_3)-\frac{M_2}{2A}, \quad K_3=M_2A,\\
J_1=N_1N_2, \quad J_2=\frac{a}{2A}(N_1N_2)-\frac{N_1}{2A}, \quad J_3=N_1A+N_3,
\end{gathered}
\tag{49}
$$

and $\bar{x}=x/2$ and $x=\gamma+\frac{\pi}{2}$. Eqs.(46)-(48) represent the first integrals of Eqs.(37) (and Eqs.(16)), and allow us to determine the aircraft's horizontal and vertical cartesian coordinates (crossrange and downrange) in the maneuver plane. The first integrals presented in Eqs.(17), (24), (36), and (46)-(48) with constants c_i, $i=1,...,6$ represent the general integral of Eq.(16). These integrals explicitly include all the aircraft maneuver parameters, such as the time, position and velocity vectors, mass, thrust, aerodynamic drag and lift, and roll angle [4]. Because of the similarity in the vehicle's models, the general integral obtained above can be used to model the trajectories of the flight vehicles, such as spacecraft, re-entry vehicles and missiles [4]- [6]. This integral has a closed-form which allows us to explicitly compute the integration constants to relate the vehicle's current state variables to its desired or terminal state variables. In this regard, any point on the trajectory can be considered as a target point and the constants can be chosen to achieve this point. Consequently, the targeting problem can be solved at any desired point thereby providing a foundation for the development and design of the targeting and guidance schemes.

Examples aircraft and re-entry vehicle dynamics models.

Given the dynamics and kinematics models of motion, and the terminal conditions, it is required to transfer a reentry vehicle (or aircraft) from its current position to the terminal configuration by satisfying a certain given criterion. Assume that the vehicle possesses nine states: position vector, $\mathbf{r}$, velocity vector, $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$. Two parameters are considered as inputs: the normalized lift coefficient, $\lambda = C_L/C_L^*$ and bank angle, $\varphi(=\sigma)$. There exist two other physical parameters: ballistic coefficient, $\beta_m = m/C_{D_0}S$, and maximum of lift-to-drag ratio, $(C_L/C_D)_{max}$. To construct the simplified dynamics model of the re-entry vehicle, it is necessary to assume a newtonian gravity field, the exponential atmospheric density and non-rotating planet [7], [1]. The dynamics model can be constructed based on the nonlinear aircraft tracking problem [1]:

$$\frac{d\ddot{\mathbf{r}}}{dt} = [\boldsymbol{\omega} \times (\ddot{\mathbf{r}} - \mathbf{g})] + [\dot{\varphi}\mathbf{e}_1^w \times (\ddot{\mathbf{r}} - \mathbf{g})] - [\dot{D}\mathbf{e}_1^w] + \dot{L}[-\mathbf{e}_2^w \sin\phi + \mathbf{e}_3^w \cos\phi] - \frac{\mu}{r^3}\dot{r}\left[\mathbf{e}_1^w - \left(\frac{\mathbf{r}}{r} \odot \mathbf{e}_1^w\right)\frac{\mathbf{r}}{r}\right], \tag{50}$$

where

$$\boldsymbol{\omega}^w = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{\dot{r}^2}, \quad \dot{L} = \frac{2}{\beta_m}\left(\frac{C_L}{C_D}\right)_{max} q(r,\dot{r})\left[\dot{\lambda} + \lambda\left(\frac{2\ddot{\mathbf{r}} \odot \dot{\mathbf{r}}}{\dot{r}^2} - \frac{\dot{\mathbf{r}} \odot \mathbf{r}}{H_0 r}\right)\right],$$

$$q(r,\dot{r}) = \frac{1}{2}\rho(r)\dot{r}^2, \quad \dot{D} = \frac{1}{\beta_m} q(r,\dot{r}) \left[2\lambda\dot{\lambda} + (1+\lambda^2) \left(\frac{2\ddot{\mathbf{r}} \odot \dot{\mathbf{r}}}{\dot{r}^2} - \frac{\dot{\mathbf{r}} \odot \mathbf{r}}{H_0 r} \right) \right],$$

$\boldsymbol{\omega}^w$ is the vector, the magnitude of which measures the amount of turning in the maneuver plane, L and D are aerodynamic lift and drag, ϕ is the bank angle. The unit vectors $\mathbf{e}_1^w, \mathbf{e}_2^w, \mathbf{e}_3^w$ denote the directions of axes of the wind frame and defined as follows [7]:

$$\mathbf{e}_1^w = \frac{\dot{\mathbf{r}} - \boldsymbol{\Omega} \times \mathbf{r}}{\dot{r}}, \quad \mathbf{e}_2^w = -\frac{\frac{\dot{\mathbf{r}}}{\dot{r}} \times \mathbf{u}_z}{||\frac{\dot{\mathbf{r}}}{\dot{r}} \times \mathbf{u}_z||}, \quad \mathbf{e}_3^w = \mathbf{e}_1^w \times \mathbf{e}_2^w,$$

where $\mathbf{u}_z = (0,0,1)^T$ is the unit vector of the Z-axis of the inertial frame. The first term in Eq.(50) represents the turning in the osculating maneuver plane defined by $\boldsymbol{\omega}^w$; the second term of this equation represents the out-of-plane motion due to rotating the lift, $\mathbf{L}$; the third term represents the drag gradient due to variations in velocity, density and induced drag due to lift. This term does not contribute to out-of-plane motion; the fourth term represents the lift magnitude variations. Out-of-plane motion can be generated by varying the lift magnitude, that is, when $\dot{L} \neq 0$; the fifth term of Eq.(50) is the gravity gradient. This term does not contribute to spiraling motion and only impacts the out-of-plane motion when the maneuver plane is not vertical. In summary, given the initial conditions, $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$ and $\ddot{\mathbf{r}}_0$ the motion is fully described if we also know the inputs $\varphi(t)$, $\lambda(t)$, the target parameters β_m and $(C_L/C_D)_{max}$. It has been shown that the dynamics model described above allows us to qualitatively describe in-plane and out-of-plane motions, to separately analyze the models such as "coordinated turn" in the maneuver plane and the spiral motion with "torsion" model which represents a rate at which the osculating maneuver plane turns as the vehicle moves along the path.

Spiraling Motion Analysis

Out-of-plane motion: "torsion" analysis.

One of the interesting models that can be developed in the context of the aircraft tracking problem is the "torsion", which is a measure of out-of-plane motion [7]:

$$\tau = \frac{\ddot{\mathbf{r}} \times \dot{\mathbf{r}}}{||\ddot{\mathbf{r}} \times \dot{\mathbf{r}}||} \odot \dddot{\mathbf{r}},$$

or

$$\tau = -\dot{\phi}\left[\frac{1}{\dot{r}} + \frac{\dot{r}}{||\ddot{\mathbf{r}} \times \dot{\mathbf{r}}||}^2 (\ddot{\mathbf{r}} - (\mathbf{e}_1^w \odot \ddot{\mathbf{r}})\mathbf{e}_1^w) \odot \mathbf{g}\right] - \dot{L}\left[\frac{\dot{r}}{||\ddot{\mathbf{r}} \times \dot{\mathbf{r}}||}^2 (\mathbf{e}_3^w \sin\phi + \mathbf{e}_2^w \cos\phi) \odot \mathbf{u}_z\right]$$

$$-3\frac{\mu}{r^3}\frac{\dot{r}^2}{||\ddot{\mathbf{r}} \times \dot{\mathbf{r}}||}^2 (\frac{\mathbf{r}}{r} \odot \mathbf{e}_1^w)\left[(\frac{\mathbf{r}}{r} \odot \times \mathbf{r}) \odot \mathbf{e}_1^w\right]. \quad (51)$$

The analysis of the "torsion" model in Eq.(51) have revealed many interesting qualitative characteristics of the motion. The spiraling frequency can be given as

$$\tau = -\dot{\phi} - \frac{1}{k^2}\frac{1}{\dot{r}^2}\left[\dot{\phi}[\ddot{\mathbf{r}} - (\mathbf{e}_1^w \odot \ddot{\mathbf{r}})\mathbf{e}_1^w] \odot \mathbf{g}\right] - \dot{L}(\mathbf{e}_3^w \sin\phi + \mathbf{e}_2^w \cos\phi) \odot \mathbf{u}_z$$

$$-3\frac{\mu}{r^3}\dot{r}\left(\frac{\mathbf{r}}{r} \odot \mathbf{e}_1^w\right)\left[(\frac{\mathbf{r}}{r} \times \mathbf{r}) \odot \mathbf{e}_1^w\right]. \quad (52)$$

It can be observed from Eq.(52) that the spiral frequency is directly proportional to $\dot{\varphi}$, where the sign represents the direction of the motion; this frequency is affected by the curvature, lift variations, and gravity gradients. Based on Eq.(51) it can be shown that the peaks of torsion occur at intervals equal to the rotation period of the lift vector and occur when the lift and gravity vectors are in opposing directions. The expressions in the second and third brackets in Eq.(51) can be shown to be negligible compared to the expression in the first bracket. The latter expression can be separated into two parts: the secular and pseudo-periodic parts (see Fig.2, where τ_b, τ_d, τ_e denote the expressions given in the first, second and third brackets respectively). The behavior in τ_b can be explained by the relationship between the values of the lift and gravitational acceleration (see Fig.3). It has been shown that the spiraling characteristics of the reentry vehicle are related directly to the ratio of lift to gravity. As is shown in Fig.2, this ratio represents an abrupt change in the torsion.

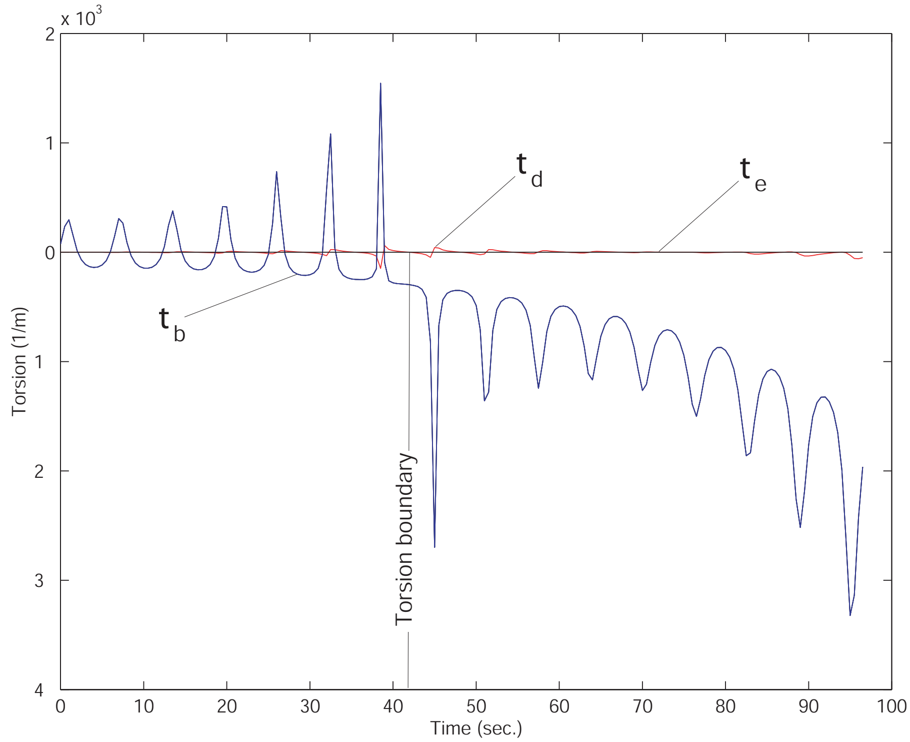

Figure 2: Interesting change in behavior observed at the "torsion boundary"

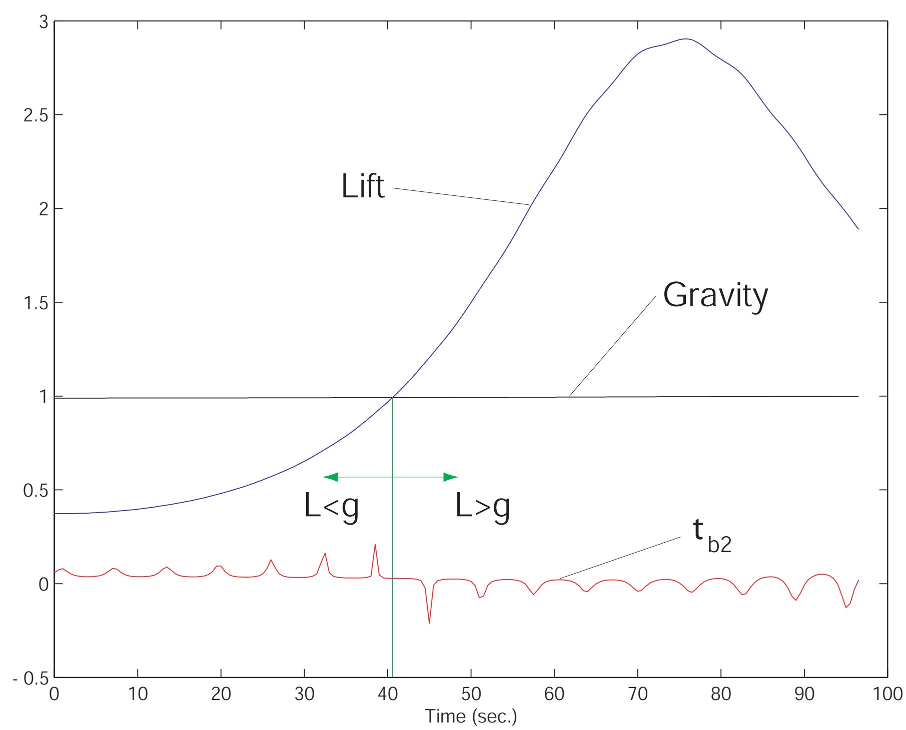

Figure 3: Behaviors of lift, gravity and pseudo-periodic component of "torsion"

In-plane motion: "coordinated turn" analysis.

If $\dot{\phi} = 0$, $\mathbf{g} = \mathbf{const}$, $\dot{L} = 0$, then the motion is confined to a plane and the corresponding model is called as the "coordinated turn model" [7]. It can be shown, the if 1) the aircraft body rate about the roll axis is zero, $\dot{\phi} = 0$, 2) the changes in lift, drag, thrust and gravitational accelerations are zero, 3) the changes in the angle of attack and sideslip angle are zero, then the resulting model describes planar trajectories in the maneuver plane. The corresponding equation of the kinematics model can be given as Eq.(1).

The analytical solutions of Eqs.(16) have revealed a direct impact of the ratio of lift to gravity, $\frac{L}{g}$ [7]. The solutions have also been validated by independent numerical integration of the equations of motion. It is expected that the analytical solutions to the "torsion" model and other models of the aircraft motion can represent more and deeper qualitative properties of motion. The proposed research includes the investigation of the models shown in Eqs.(50), (51) and (52), and the development of corresponding analytical solutions and design of analytical real-time targeting and guidance laws for flight vehicles.

Conclusions

The explicit, closed-form, analytical solutions of the aircraft's equations for a non-steady flight in a vertical plane have been obtained. These solutions are based on the six independent first integrals for heading angle, magnitude of velocity vector, time, and three position vector components. All integrals are expressed in elementary and transcendental functions in terms of the flight path angle. Using the assumptions of the paper, it has been shown that these equations are equivalent to the third order vector differential equation of flight in a maneuver plane which was used in the development of the geometric nonlinear filter. Due to the generality of the flight equations used in the paper, the analytical solutions obtained here can be used in the design of trajectories of aircraft, spacecraft, re-entry vehicles and missiles within the conditions of subsonic speeds. The completeness and explicitness of the closed-form analytical solutions with respect to the independent variable is the main advantage of the results presented in this paper. These results may find potential applications in the design of on-board targeting and guidance schemes.

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