

# **Fast Gravitational Field Model using Adaptive Orthogonal Finite Element Approximation**

**Ahmad Bani Younes, Brent Macomber, Robyn M. Woollands, Austin Probe, John L. Junkins**

*Texas A & M University, Aerospace Engineering Dept, H.R. Bright, 3141 TAMU, College Station, TX, 77843-3141*

**Xiaoli Bai**

*Optimal Synthesis, Inc, 95 1<sup>st</sup> St. Suite 240, Los Altos, CA 94022*

## **ABSTRACT**

Recent research has addressed the increasing computational challenge that the current state of the art global gravity expansions involve tens of thousands of terms in a theoretically infinite order expansion (some spherical harmonic gravity models extend to degree and order 200 with over 30,000 terms). As global gravity models become more detailed and expensive, and since, acceleration must be computed at numerous local points to generate high precision orbits), and finally, with the advent of >20,000 objects whose orbits must be propagated for Space Situational Awareness, the expense of gravity computation has emerged as a vitally important computational challenge. In this paper we consider orthogonal approximation methods to establish an FEM high accuracy global gravity field representation. The FEM model replaces the global spherical harmonic series with a family of locally precise orthogonal polynomial approximations for efficient computation. Our preliminary results showed that CPU time to compute the state of the art (degree and order 200) spherical harmonic gravity is reduced by 4 to 5 orders of magnitude while maintaining > 9 digits of accuracy. Most of the speedup is due to adopting the orthogonal FEM approach, but a radial adaptation method to modify the approximation degree is introduced that results in an additional order of magnitude speedup. The Adaptive Orthogonal Finite Element Gravity Model (AOFEGM) has wide applicability to establish a new generation of efficient trajectory propagation algorithms. For example, when used in conjunction with the orthogonal Finite Element Model (FEM) gravity approximations discussed herein, the highly parallelizable Chebyshev-Picard path approximation enables truly revolutionary speedups in orbit propagation without accuracy loss.

## **1. INTRODUCTION**

There are several treatments of discrete approximation using Chebyshev polynomials [1-10]. Among the more comprehensive of these are the texts [5,7]. In [4], orthogonal approximation is placed in a broader context of multi-resolution approximation via linear and nonlinear input/output maps. Several applications of the orthogonal approximation are considered in Astrodynamics. This paper establishes highly efficient local approximation of high degree and order geopotential models, replacing the global spherical harmonic series with a family of locally precise orthogonal polynomial approximations for efficient computation. We introduce a method that adapts the approximation degree radially to ensure that the maximum acceleration error  $< 10^{-9}$  m/s<sup>2</sup>. This method takes into account the fact that the highest degree approximations are required near the Earth's surface, whereas lower degree approximations are required as we move further away from the Earth and the radius increases. Specifically, we replace the GRACE [11,12] (156, 156) spherical harmonic model with a global family of local orthogonal polynomial approximations. We also replace the (200, 200) Earth Gravitational Model EGM 2008 [13], using the same FEM orthogonal approximation approach to observe the dependence of the relative computational advantage on higher order gravity terms.

The classical solution to Laplace's equation for gravity is adopted using the globally valid spherical harmonic gravity potential model, where the spherical harmonic (SH) approach is slow and reveals the three main challenges [11,14,15-18]: (1) Choosing a finite upper limit of the series defines the accuracy (the more we know about gravity, the more terms are required and the more it costs to compute acceleration), (2) Convergence is very inefficient and slow for  $n > 2$ , so, for the current state of the art, tens of thousands of terms are required to obtain a sufficiently high accuracy global gravity representation, (3) The North and South poles represent non-free singularities of the usual spherical coordinates. In view of the slow convergence of global gravity models, we are motivated to truncate the

classical expansion at  $n = 2$  and introduce a finite element model (FEM) local gravity representation of the higher order perturbation in the anticipation that much lower degree locally valid functions can be used to efficiently model and compute local gravity perturbations. Applicable to both irregular and near-spherical shaped bodies, methods in this class expedite computations by effectively trading computer memory for runtime speed. First proposed by Junkins in 1976 [11], geopotential FEM interpolation methods have been bolstered recently by the extraordinary memory resources of common computers. A variety of approximation techniques and basis functions have been employed for gravity field representation, including weighting functions [3,14,19,20,21], wavelets [22], splines [22,23], octrees [24], psuedocenters [25] and 3D digital modeling [26]. Each interpolation method balances accuracy with efforts to minimize runtime speed and memory footprint cost while achieving exactness, continuity and smoothness as appropriate.

## 2. ORTHOGONAL APPROXIMATION

We unify and extend classical results from function approximation theory and consider their utility in Astrodynamics. Least square approximation, using the classical Chebyshev polynomials as basis functions, is reviewed for discrete samples of the to-be-approximated function. We extend the orthogonal approximation ideas to  $n$ -dimensions in a novel way, through the use of array algebra and Kronecker operations. Approximation of test functions illustrates the resulting algorithms and provides insight into the errors of approximation, as well as the associated errors arising when the approximations are differentiated or integrated. We first review classical discrete polynomial approximation results for one and two dimensions and introduce a convenient array algebra means to extend the one dimensional orthogonality results to higher dimensions. This path avoids the curse of dimensionality and establishes the results needed for efficient computation. Several simple examples are provided to enable the efficacy and utility of the methodology to be appreciated heuristically.

Let us first set the context by considering the approximation of a single-valued function of one independent variable

$$g(x), \{x_{\min} \leq x \leq x_{\max}\} \quad (1)$$

To put the problem in a non-dimensional framework, we first introduce a new independent variable  $\xi$  such that  $\{-1 \leq \xi \leq 1\}$ . It is easy to verify the forward and inverse transformations:

$$\xi(x) = 2(x - x_{\min}) / (x_{\max} - x_{\min}) - 1, \text{ and } x(\xi) = x_{\min} + (\xi + 1)(x_{\max} - x_{\min}) / 2 \quad (2)$$

Substituting the second of Eqs (2) into Eq (1), we see that we wish to approximate the function

$$f(\xi) = g(x(\xi)) \equiv g(x_{\min} + (\xi + 1)(x_{\max} - x_{\min}) / 2) \quad (3)$$

In the case of general basis functions, we seek to approximate  $f(\xi)$  as a linear combination of a prescribed set of  $N+1$  linearly independent basis functions  $\{\phi_0(\xi), \phi_1(\xi), \dots, \phi_N(\xi)\}$  as

$$f(\xi) \cong \sum_{n=0}^N a_n \phi_n(\xi) \quad (4)$$

For the case of discrete measurement samples, we introduce a set of sample points (nodes) as  $\{\xi_0, \xi_1, \dots, \xi_M; M \geq N\}$ ; the residual approximation error at each measurement node is

$$r_j = f(\xi_j) - \sum_{n=0}^N a_n \phi_n(\xi_j); \quad j = 0, 1, \dots, M \quad (5)$$

or in vector-matrix notation

$$\mathbf{r} = \mathbf{f} - \Phi \mathbf{a} \quad (6)$$

where

$$\mathbf{f} = \begin{bmatrix} f(\xi_0) \\ f(\xi_1) \\ \vdots \\ f(\xi_M) \end{bmatrix}, \quad \Phi = \begin{bmatrix} \phi_0(\xi_0) & \phi_1(\xi_0) & \cdots & \phi_N(\xi_0) \\ \phi_0(\xi_1) & \phi_1(\xi_1) & \cdots & \phi_N(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(\xi_M) & \phi_1(\xi_M) & \cdots & \phi_N(\xi_M) \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \quad (7)$$

The method of least squares seeks the coefficient vector ( $\mathbf{a}$ ) that minimizes the weighted sum square of the residuals

$$J = \frac{1}{2}(\mathbf{f} - \Phi\mathbf{a})^T W(\mathbf{f} - \Phi\mathbf{a}); \quad W = W^T \text{ (positive definite weight matrix)} \quad (8)$$

It follows [9] that the least square minimization solution for  $\mathbf{a}$  leads to the *normal equations*

$$\mathbf{a} = (\Phi^T W \Phi)^{-1} \Phi^T W \mathbf{f} \quad (9)$$

Restricting  $W$  to be diagonal hereinafter, and choosing a special class of orthogonal basis functions,  $\Phi^T W \Phi$  can be rendered a diagonal matrix so the matrix inverse in Eq (9) is trivial. The orthogonality conditions depend jointly on the set of basis functions, the set of node locations and the weight matrix (more generally,  $W = W^T$  may be fully populated). For the case that the above orthogonality conditions are satisfied, the explicit solution for the coefficients of Eq (9) is given by the independent/uncoupled ratios of inner products as

$$a_\alpha = \frac{\langle \phi_\alpha(\xi), f(\xi) \rangle}{\langle \phi_\alpha(\xi), \phi_\alpha(\xi) \rangle} \equiv \frac{\sum_{j=0}^M W_j \phi_\alpha(\xi_j) f(\xi_j)}{\sum_{j=0}^M W_j \phi_\alpha^2(\xi_j)} \equiv \frac{1}{c_\alpha} \sum_{j=0}^M W_j \phi_\alpha(\xi_j) f(\xi_j), \text{ for } \alpha = 0, 1, 2, \dots, N \quad (10)$$

An important special case arises when we make a specific choice of orthogonal basis functions, namely  $\{\phi_0(\xi), \phi_1(\xi), \dots, \phi_N(\xi)\} = \{T_0(\xi), T_1(\xi), \dots, T_N(\xi)\}$ , i.e., we choose the classical Chebyshev polynomials  $\{T_0(\xi), T_1(\xi), \dots, T_N(\xi)\}$ , as discussed in references [1,2,4,5,20], as the basis functions and also we choose the  $N+1$  cosine sample points (also known [1,2,4,5,20] as the CGL nodes in honor of Chebyshev-Gauss-Lobatto):

$$\xi_j = -\cos(j\pi / M), \quad j = 0, 1, 2, \dots, M \quad (11)$$

If a vector-matrix form is desired for the least squares solution for the coefficients, we can re-arrange Eqs (9) in the form shown below, after utilizing the orthogonal condition

$$\mathbf{a} = A\mathbf{f} \quad (12)$$

where the Chebyshev least square operator matrix is

$$A = \frac{1}{M} \begin{bmatrix} T_0(\xi_0)/2 & T_0(\xi_1) & \cdots & T_0(\xi_{M-1}) & T_0(\xi_M)/2 \\ T_1(\xi_0) & 2T_1(\xi_1) & \cdots & 2T_1(\xi_{M-1}) & T_1(\xi_M) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{N-1}(\xi_0) & 2T_{N-1}(\xi_1) & \cdots & 2T_{N-1}(\xi_{M-1}) & T_{N-1}(\xi_M) \\ T_N(\xi_0) & 2T_N(\xi_1) & \cdots & 2T_N(\xi_{M-1}) & T_N(\xi_M) \end{bmatrix} \quad (13)$$

Note that the particular weight matrix  $W = \text{diag}\{\frac{1}{2}, 1, 1, \dots, 1, 1, \frac{1}{2}\}$  can be shown to be consistent with the classical Chebyshev polynomials satisfying the orthogonality conditions of Eq (10). The choice of an identity matrix, for example, together with the Gramm-Schmidt process [1,2], gives rise to a related set of orthogonal polynomials. The approximation properties of the Chebyshev polynomials are well-researched and a substantial literature exists related to this choice, therefore we adopt the slight modification of the identity weight matrix. We believe the above formulation leads to a logical path to generalize the classical weighted least square formulations to the analogous developments for approximating functions of  $n$  variables, as we show below.

The Chebyshev polynomial formulation is known to be relatively immune to the so-called Runge Phenomena wherein the approximation errors near the end of the data support at  $\pm 1$  can become unacceptably large. The dense sampling near the ends of the approximation interval implicitly reduces errors near the boundary. Also, the fact that no numerical matrix inversion is required for orthogonal polynomials means that approximation can be robustly computed at any desired or required order. These advantages were illustrated by performing several test examples, and can be found in Bani Younes [1]. The flowcharts explaining the 1-D, 2-D, and  $n$ -D approximations are shown in Figure 1, Figure 2 and Figure 3 respectively.

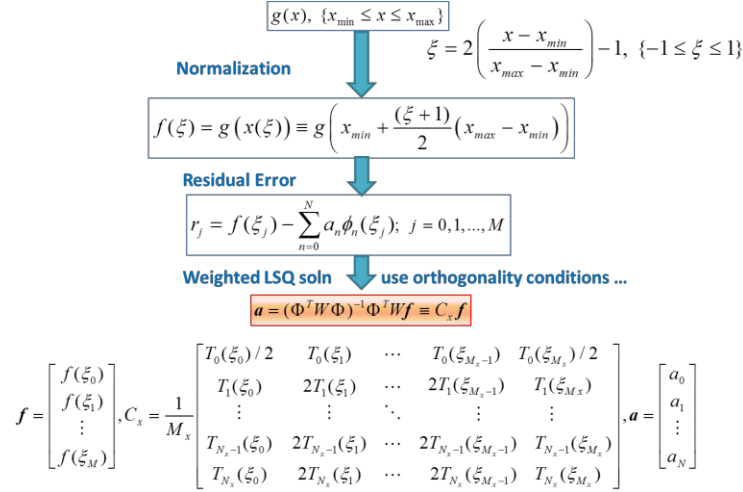


Figure 1: 1-D Approximation.

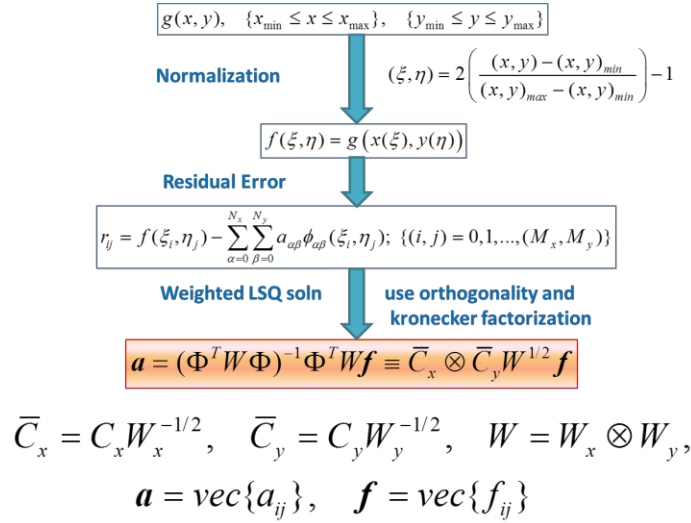


Figure 2: 2-D Approximation.

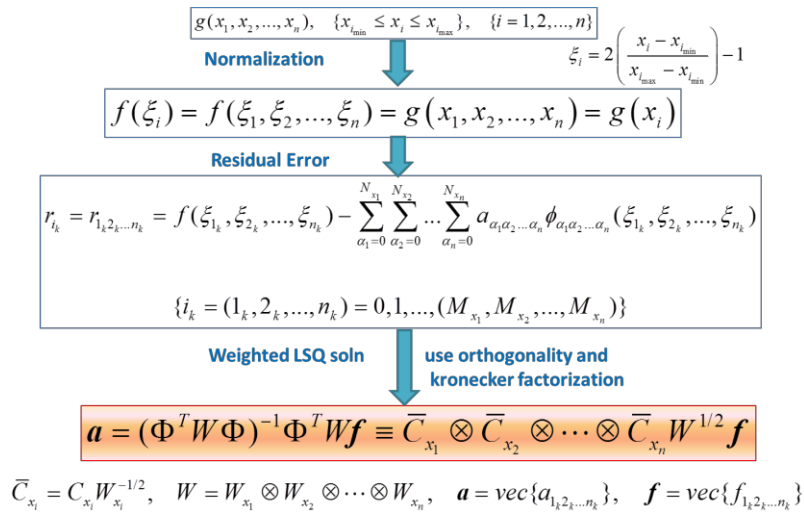


Figure 3: n-D Approximation.

### 3. ORTHOGONAL FINITE ELEMENT REPRESENTATIONS OF THE GEOPOTENTIAL

We first consider the construction of an orthogonal FEM approximation to the gravity potential field model determined from the Gravity Recovery And Climate Experiment (GRACE). The GRACE Gravity Model has been publicly released [3,19,10]. Access to the model's coefficients and other descriptive files about GRACE were obtained from [19,10]. In view of the slow convergence of global gravity models, we are motivated to introduce finite element model (FEM) local gravity representations in the anticipation that much lower degree functions can be used to efficiently model and compute local gravity. The literature on this subject initiated with our classical developments [11-13] and has recently been explored by other others [15-17]. In our developments herein, we have solved a key historical challenge implicit in this class of methods for geopotential representation: How do we structure the FEM models to render them radially adaptive and efficient, so that the resulting algorithms “automatically know” about the rapid radial decay of the high frequency terms and more to the point, which terms in the FEM representation to retain, as a function (mainly) of radial distance from geocenter. We consider the total gravity potential model split into reference and disturbance gravity terms, where, for example, the global reference gravity term includes the  $O(1)$  2-body and the  $O(10^{-3})$   $J_2$  oblateness terms whereas “everything else” (all the higher degree and order terms) are considered perturbative gravity disturbance to the reference model. The potential and the acceleration are

$$\begin{aligned}
 \underbrace{U(r, \lambda, \phi)}_{\text{Total}} &= \underbrace{U_{2B}(r, \lambda, \phi) + U_{J_2}(r, \lambda, \phi)}_{\text{Reference}} + \underbrace{\Delta U(r, \lambda, \phi)}_{\text{Disturbance}} \\
 \ddot{\mathbf{r}} &= -\frac{\partial U_{\text{Total}}}{\partial \mathbf{r}} = -\underbrace{\frac{\partial}{\partial \mathbf{r}} \frac{GM}{r} \left\{ 1 - \frac{3}{2} J_2 \left( \frac{R_{\oplus}}{r} \right)^2 (3 \sin^2(\phi) - 1) \right\}}_{\text{Reference}} - \underbrace{\frac{\partial \{\Delta U\}}{\partial \mathbf{r}}}_{\text{Disturbance}}
 \end{aligned} \tag{14}$$

The preliminary results show that, for routine exo-atmosphere orbit calculations, the FEM gravity model can be computed to an accuracy of 9 to 10 digits with a 4 order of magnitude reduction in CPU time, relative to using the correspondingly accurate spherical harmonic representation. Radial adaptation of the FEM computation can be readily implemented by a one-time a priori computational process (at the time the FEM model is established) to find the maximum degree to maintain a prescribed accuracy tolerance, as a function of radial displacement through each FEM element. A  $\sim 4$  order of magnitude computational speedup by this approach can be even further enhanced by introduction of parallelization in conjunction with the Chebyshev-Picard methods [1,2] discussed below where many gravitational acceleration evaluations at judicious nodal points along a known approximate path can be simultaneously computed in an iterative path approximation algorithm. In most cases, these path approximations are found to converge over 2 to 3 orbits and therefore allow 2 or more orders magnitude additional speedup. Using a fusion of adaptive orthogonal FEM gravity approximation and the Chebyshev-Picard orbit path approximation methods, we can achieve supercomputer orbit computation performance with a desktop computer.

As a specific example FEM grid, a sphere of radius  $R$  is covered by a 2-D mesh (4x4) degree  $(\lambda, \phi) : 0 \leq \lambda \leq 360; -88 \leq \phi \leq 88$  cellular grid, except for the polar caps of angular radius 2 degrees. At arbitrary  $r \{r_{\min} = R_{\oplus} \leq r \leq r_{\max} = R_{\oplus}\}$ , a large family of spherical shells is sampled using the cosine distribution as shown in Figure 4.

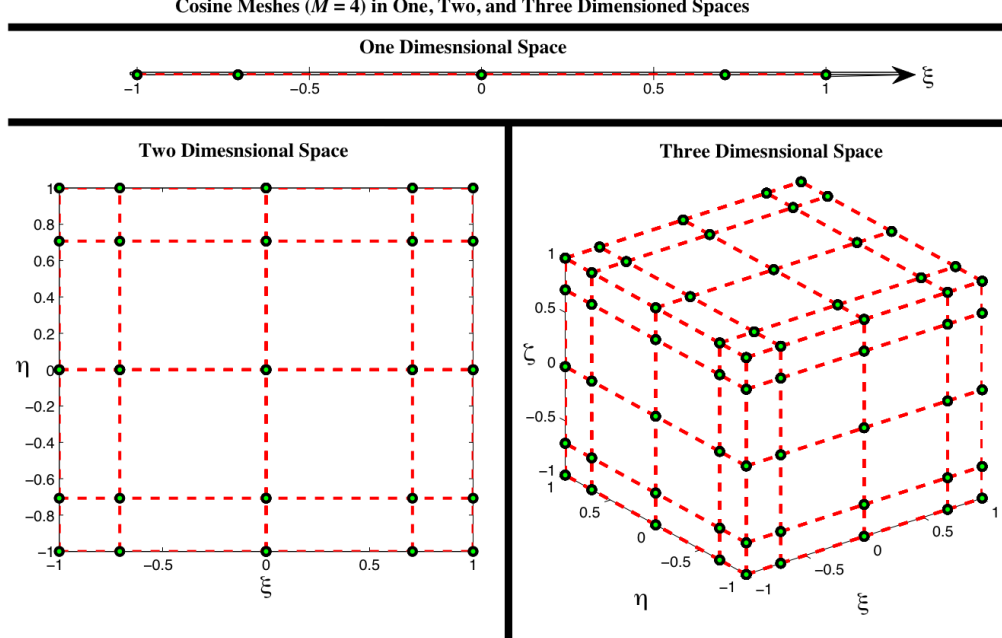


Figure 4: Multidimensional Cosine Meshes for Discrete Orthogonality Chebyshev Polynomials.

Let the gravity data  $U(r = \text{constant}, \lambda, \phi)$  on a given spherical shell be transformed into  $U(\xi = \text{constant}, \zeta_i, \eta_j)$  where  $-1 \leq \zeta_i, \eta_j \leq +1$  with the sample points located according to the cosine distribution. The transformed position ( $r$ ) obeys smart “cosine-like” transformation as a function of the transformed radial variable  $\xi$ :

$$r = r_{\min} + (r_{\max} - r_{\min}) \left[ 1 - \cos\left(\frac{\pi}{4}(1 + \xi)\right) \right], \left\{ \begin{array}{l} -1 \leq \xi \leq +1 \\ r_{\min} = R_{\oplus} \leq r \leq r_{\max} = 7R_{\oplus} \end{array} \right\} \quad (15)$$

where  $\xi_j = -\cos(j\pi / M)$ ,  $j = 0, 1, 2, \dots, M$ . This transformation is intended, for uniform samples, to generate higher density  $r$  samples on the left (near  $r_{\min}$ ) and less dense samples on the right (near  $r_{\max}$ ) compared with the classical cosine sampling, see Figure 5 (a). The dense sampling near  $r_{\min}$  ensures that a higher density of measurement samples is considered where the gravity perturbations are maximum and have the largest local significant differential changes. Since the gravity anomalies “die out” rapidly with increasing radius, less dense sampling is anticipated for increasing  $r$ , see Figure 5 (b).

It is important to determine the required polynomial order, as a function of radius, to adaptively maintain an approximation error tolerance. Radial order adaptation enables enormous speedups in the computation of the state of the art gravity models. For this insight, the required acceleration error tolerance is determined as a function of the polynomial order  $N$ . For instance, a (4x4) degree square area at the Earth's surface is sampled using 2D Chebyshev distribution. The convergence error is defined by the maximum absolute error between the truth model and the approximation acceleration. We selected a conservative maximum approximation error of  $10^{-9}$  m/s<sup>2</sup> as the tolerance for errors in replacing the high degree and order gravity model by FEM approximation. Figure 6 below shows maximum absolute error of the approximated “disturbance” acceleration (x, y, and z components) as function of Chebyshev polynomial order  $N$ . In this case, the reference gravity potential is simply the point mass term and the  $J_2$  perturbation, everything else approximated and plotted in Figure 7 and Figure 8. The (156x156) GRACE model is adopted as the truth, and the approximation errors are reduced by adjusting the degree of the Chebyshev polynomials to achieve convergence once the error falls below the desired tolerance.

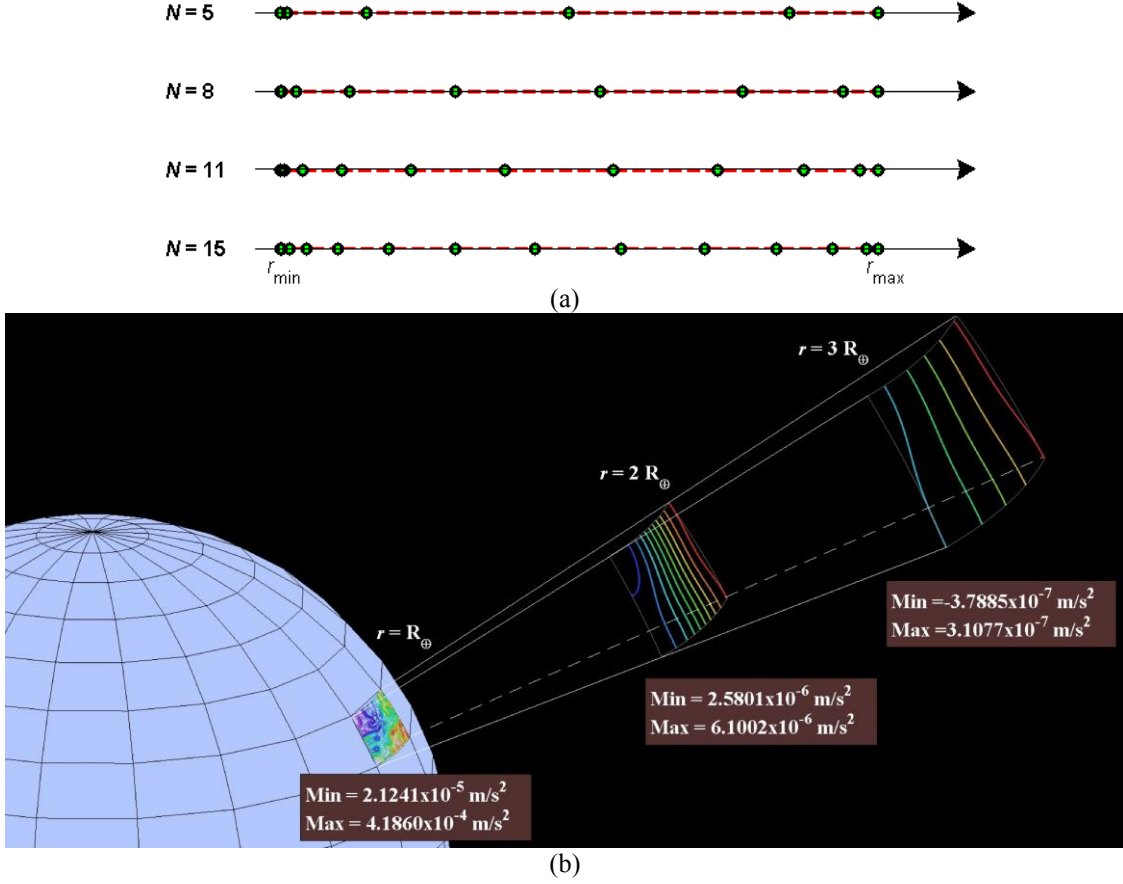


Figure 5: (a) Cosine-like Sampling for the Radial Distance, Note the Density of the Nodes is Highest Near  $r_{\min}$ . (b) Radial Gravity (EGM2008 200x200) Contoured on Three Spherical Shells.

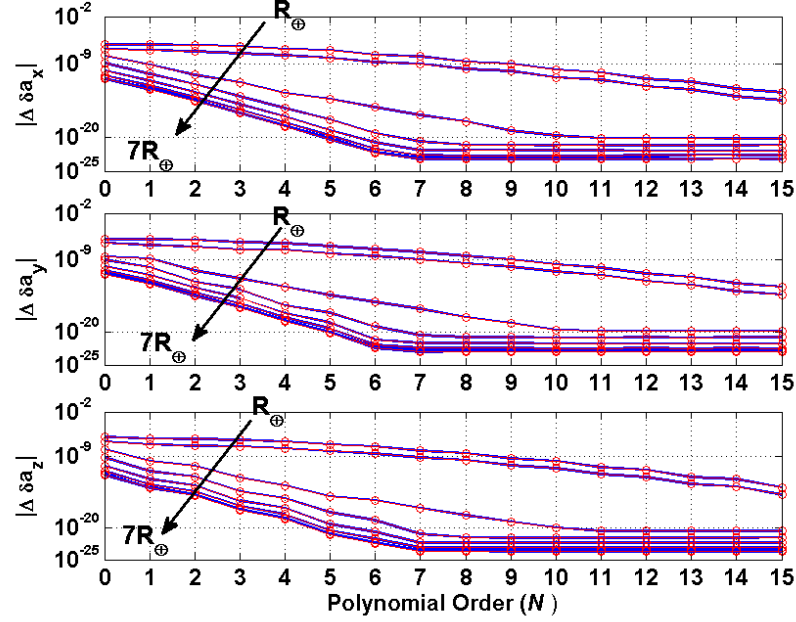


Figure 6: Maximum Error of Chebyshev FEM Gravity FEM gravity Approximation ( $\text{m/s}^2$ ) as a function of Polynomial Order  $N$ , for Various Radial Distances, GRACE 156x156.



At the Earth's surface, with the (4x4) degree FEM cell size, this is achievable at  $N = 10$ , whereas at  $r_{\max}$ , the maximum allowable error is achieved with only a first degree ( $N = 1$ ) model for the gravity disturbance acceleration (as an additive local correction to the reference global model). This indicates that the local gravity perturbation potential at the Earth's surface is approximated by 121 orthogonal polynomial terms, whereas only a linear approximation of local disturbance acceleration is required at  $r_{\max}$ . It is to be expected that the required polynomial order decreases monotonically as we move away from the Earth's surface  $r_{\min} = R_{\oplus}$  out to  $r_{\max} = 7 R_{\oplus}$ , outside of the GEO radius; we found that only first degree polynomials are required at  $r_{\max}$ , and thus only 4 polynomial terms are needed for all three components of acceleration. Since, the gravity field within the range of interest  $[R_{\oplus}$  up to  $7 R_{\oplus}]$  has significant variation as  $r$  varies; it becomes useful to divide the model into two concentric spherical shell regions for the sake of FEM representation:

Region I: Atmospheric region  $r \in [R_{\oplus}, 1.02R_{\oplus}]$ ,

Region II: The mostly exo-atmospheric region  $r \in [1.02R_{\oplus}, 7R_{\oplus}]$ .

On a serial machine, the FEM approach is 2 orders of magnitude more computationally efficient at the Earth's surface than the (156x156) spherical harmonic expansion, and due to the radial adaptation feature of this approach, the FEM computational cost is, remarkably, reduced an additional  $\sim 2$  orders of magnitude for  $1.02 R_{\oplus} < r < 7 R_{\oplus}$ .

This implies that for routine exo-atmosphere orbit calculations, the FEM gravity model is computed with a 9 to 10 digit accuracy and a 4 order of magnitude reduction in CPU time, relative to using the correspondingly accurate spherical harmonic representation. Radial adaptation of the FEM computation is readily implemented by a one-time a priori computational process

Figure 7 : Radial Perturbative FEM Gravity Approximation at the Earth's Surface (m/s<sup>2</sup>).

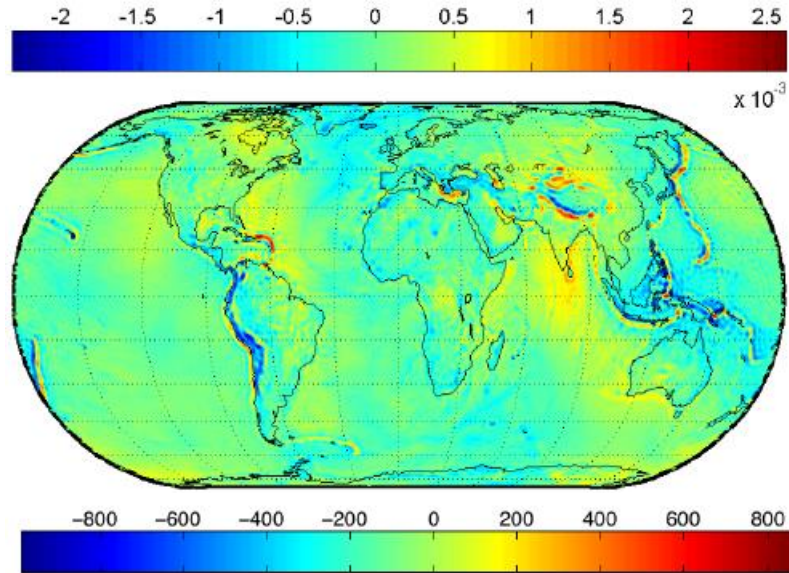


Figure 8: Global FEM Gravity Potential Approximation at the Earth's Surface (m<sup>2</sup>/s<sup>2</sup>).

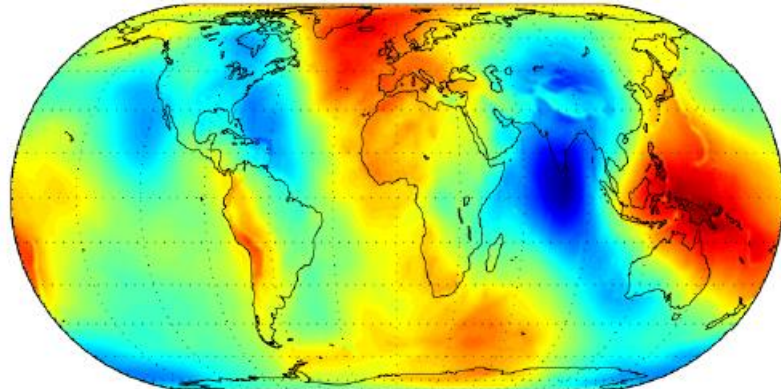




Figure 9 shows the polynomial order required to achieve the maximum approximation error ( $\text{m}^2/\text{s}^2$ ) as a function of spherical shell radius  $r$ , over each of the two regions. For Region I, the maximum polynomial order required corresponds to the shell closest to the Earth's surface, and similarly in Region II the maximum polynomial order  $N = N_{\text{max}} = 7$  also matches the shell nearest to the Earth. However, a much smaller  $N$  is required at large  $r$ . To compute acceleration from the FEM model, we use the maximum polynomial order that ensures consistency with the full polynomial model, but retain for orbit computation only the  $N_{\text{reqd}}(r) < N_{\text{max}}$  terms that contribute at that particular  $r$ . So for larger  $r$ , an additional order of magnitude computational speedup is achieved by including only the significantly non-zero terms. The inherent cosine sampling of the radial position allows us to use the standard equations for orthogonal least square Chebyshev approximation to obtain the polynomial coefficients.

The computational speed of the FEM compared with that of the GRACE (156, 156) [11,12] model favors the highly accurate FEM approximation by about four orders of magnitude. Figure 10 displays a more detailed computational comparison between the two gravity representations. It is clear that the computation time for the FEM decreases as the required polynomial order decreases.

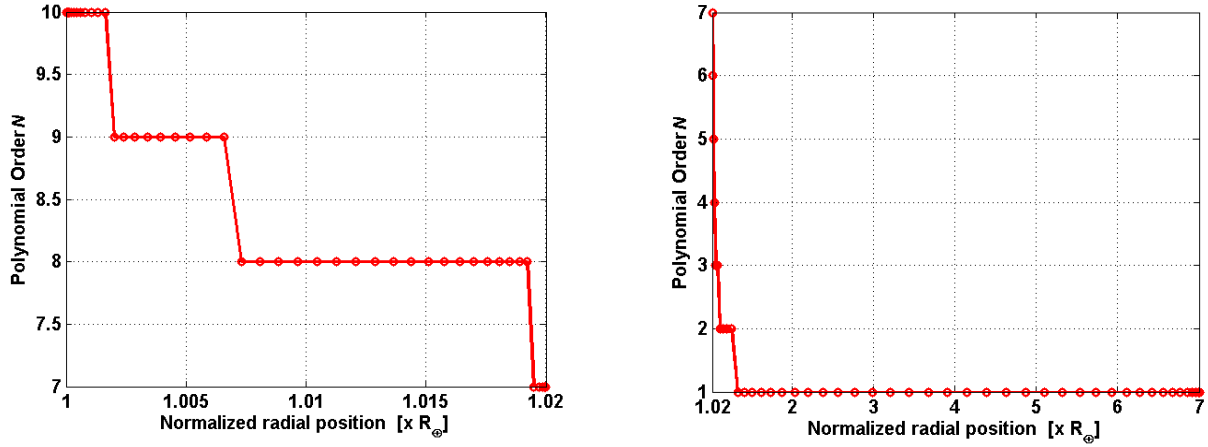


Figure 9: Polynomial Order  $N$  versus  $r$ , GRACE 156x156.

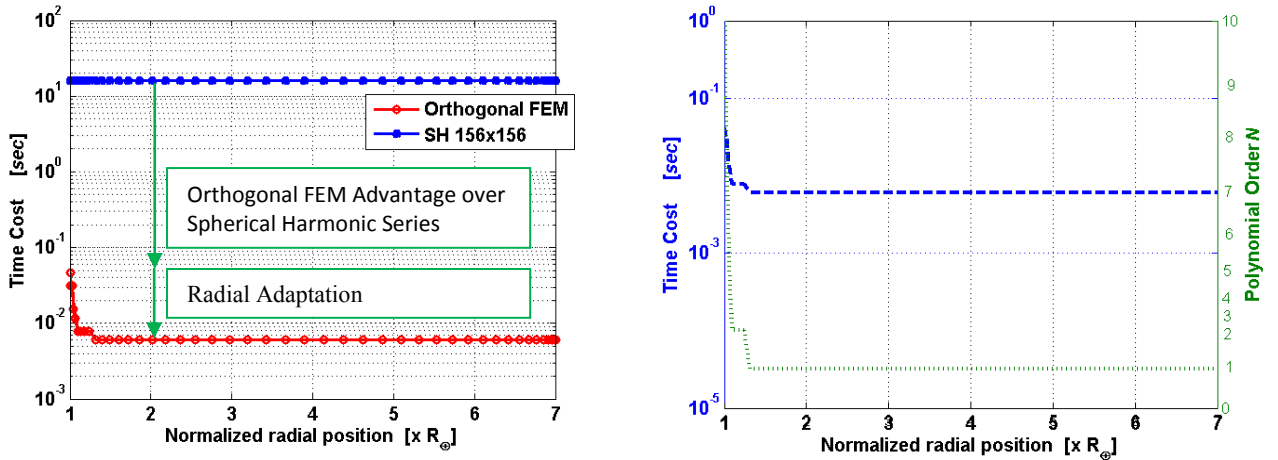


Figure 10: Computation Speed of the FEM versus the Spherical Harmonic, GRACE 156x156.

#### 4. CONCLUSION

The classical spherical harmonic expansion and analogous global models require  $> 10^5$  terms in a series to compute  $> 9$  digit converged local acceleration with a single global expansion. This is a challenge and it is therefore not attractive to utilize these high order global models to compute local gravity for the purpose of efficient and accurate trajectory computation. In this paper we have introduced our method of adaptive finite element gravitational approximation. This automatically determines the minimum number of terms required for the approximation as a function of radial distance from the Earth, but still maintaining the prescribed accuracy. In addition, we have

demonstrated that a four order of magnitude speedup in the computation time is achievable with the FEM compared with spherical harmonic series computation. These significant results have a strongly positive impact on the field of orbital trajectory propagation and we look to seeing the benefits of this implementation on actual flights.

## 5. REFERENCES

1. Bani Younes, A., Orthogonal Polynomial Approximation in Higher Dimensions: Applications in Astrodynamics., Ph.D. dissertation, Texas A&M Univ, College Station, TX, 2013.
2. Bai, X., Modified Chebyshev-Picard Iteration Methods for Solution of Initial Value and Boundary Value Problems., Ph.D. dissertation, Texas A&M Univ, College Station, TX, 2010.
3. Junkins, J., Miller, G., and Jancaitis, J., "Weighting Function Approach to Modeling of Irregular Surface," *Journal of Geophysical Research*, Vol. 78, No. 11, 1973, pp. 1794-1803.
4. Singla, P. and Junkins, J. L., *Multi-resolution Methods for Modeling and Control of Dynamical Systems*, CRC Press, 2009.
5. Mason, J. and Handscomb, D., *Chebyshev Polynomials*, Chapman and Hall/CRC, 2003.
6. Chebyshev, P. L., "Thorie De Mcanismes Connus Sous Le Nom De Paralllogrammes," *Memoires des Savants etrangers pr\_sent\_es\_a l'Acad\_emie de Saint- Petersburg*, Vol. 7, 1857, pp. 539-586.
7. Fox, L. and Parker, I. B., *Chebyshev Polynomials in Numerical Analysis*, London, UK: Oxford University Press, 1972.
8. Snay, R. A., "Applicability of Array Algebra," *Rev. Geophys.*, Vol. 16, No. 3, 1978, pp. 459-464.
9. Schaub, H. and Junkins, J. L., *Analytical Mechanics of Space Systems*, Reston, VA, AIAA Education Series, 2nd ed., 2009.
10. Battin, R., *An Introduction to the Mathematics and Methods of Astrodynamics*, Reston, VA: American Institute of Aeronautics and Astronautics, Inc, revised ed., 1999.
11. GGM03, "Gravity Recovery and Climate Experiment (GRACE)," <http://www.csr.utexas.edu/grace/gravity/>, Aug 16, 2011.
12. Russell, R. P., "GRACE Spherical Harmonic Coefficients," Private Communication with J Junkins, 2012.
13. EGM2008, "Earth Gravitational Model 2008 (EGM2008)," <http://Earth-info.nga.mil/GandG/wgs84/gravitymod/egm2008/>, May 06, 2013.
14. Junkins, J., "Investigation of Finite-Element Representations of the Geopotential," *AIAA Journal*, Vol. 14, No. 6, 1976, pp. 803-808.
15. Pines, S., "Uniform Representation of the Gravitational Potential and its Derivatives," *AIAA Journal*, Vol. 11, 1973, pp. 1508-1511.
16. Casotto, S. and Fantino, E., "Evaluation of Methods for Spherical Harmonic Synthesis of the Gravitational Potential and its Gradients," *Advances in Space Research*, Vol. 40, 2007, pp. 69-75.
17. Lundberg, J. B. and Schutz, B. E., "Recursion Formulas of Legendre Functions for Use with Nonsingular Geopotential Models," *Journal of Guidance Control Dynamics*, Vol. 11, Feb. 1988, pp. 31-38.
18. Tapley, B., Ries, J., Bettadpur, S., Chambers, D., Cheng, M., Condi, F., and Poole, S., "The GGM03 Mean Earth Gravity Model from GRACE," *AGU Fall Meeting Abstracts*, Dec. 2007, pp. A3.
19. Arora, N. and Russell, R., "Fast, Efficient, and Adaptive Interpolation of the Geopotential," *AAS/AIAA Astrodynamics Specialist Conference*, Vol. AAS 11-501, Girdwood, AK, 2011.
20. Junkins, J. and Jancaitis, J., "Modeling in n-Dimensions Using a Weighting Function Approach," *Journal of Geophysical Research*, Vol. 79, No. 23, 1974, pp. 3361-3366.
21. Junkins, J. L. and Engels, R. C., "Local Representation of the Geopotential by Weighted Orthonormal Polynomials," *Journal of Guidance, Control, and Dynamics*, Vol. 3, No. 1, 1980, pp. 55-61.
22. Beylkin, G. and Cramer, R., "Toward Multiresolution Estimation and Efficient Representation of Gravitational Fields," *Celestial Mechanics and Dynamical Astronomy*, Vol. 84, Sept. 2002, pp. 87-104.
23. Lekien, F. and Marsden, J., "Tricubic Interpolation in Three Dimensions," *Journal of Numerical Methods and Engineering*, Vol. 63, 2005, pp. 45-471.
24. Colombi, A., Hirani, A. N., and Villac, B. F., "Adaptive Gravitational Force Representation for Fast Trajectory Propagation Near Small Bodies," *Journal of Guidance Control and Dynamics*, Vol. 31, 2008, pp. 1041-1051.
25. Hujsak, R. S., "Gravity Acceleration Approximation Functions," *Advances in the Astronautical Sciences*, Vol. 93, 1996, pp. 335-349.
26. Oltrogge, D., "AstroHD: Astrodynamics Modeling With a Distinctly Digital Flavor," *AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, Honolulu, HI, August 18-21 2008.