Minimum-Time, Constant-Thrust Orbit Transfers with Noncircular Boundary Conditions

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It has been shown previously that the accumulated velocity change of a spacecraft under continuous thrust is nearly constant over a wide range of thrust magnitudes for multi-revolution, time-optimal orbit transfers, approximately equal to the difference in initial and final circular velocities. This same behavior exists for multi-revolution transfers between noncircular orbits, but the value of the accumulated velocity change must be determined numerically. For continuous-thrust transfers of less than one revolution, the approximate initial Lagrange costates and final flight times for minimum-time orbit transfers have been shown to reduce to simple algebraic expressions which may be used to initialize continuation methods. If the smaller of either the initial or final orbits is noncircular, chaotic behavior may be observed using continuation methods. This paper examines chaotic discontinuities and local minima in the continuation method and offers techniques to help automate parameter searches for minimum-time, continuous-thrust orbit transfers.

INTRODUCTION

Minimum-time, continuous-thrust orbit transfers have been studied in great detail and have been shown[1] to have certain predictable characteristics when the initial and final orbits are exactly circular and coplanar. In particular, the accumulated velocity change ($Vacc$) due to the action of the propulsion system is nearly constant for a wide range of constant thrust level, multi-revolution transfers which is useful for calculating the approximate total flight time for a given initial acceleration value. In the multi-revolution case, $Vacc$ is approximately equal to the difference between the initial and final circular velocities. Since the time of flight is approximately equal to $Vacc$ divided by the magnitude of the acceleration produced by the propulsion system, this provides a simple relationship to estimate the minimum time of flight between two circular, coplanar orbits under continuous thrust.

To obtain the minimum-time control law for the instantaneous thrust angle relative to the local horizon, $\phi = \tan^{-1}(\lambda_u/\lambda_v)$, a variational Hamiltonian is formed using Lagrange costates, λ, as shown in a classical Earth-to-Mars transfer example in the literature[3]. However, the physical boundary conditions do not provide enough information to determine the optimal initial values of the Lagrange costates or the final flight time, so they must be found numerically. For continuous-thrust orbit transfers with less than one revolution, the approximate initial Lagrange costates and final flight times for minimum-time trajectories have been shown[2] to reduce to simple algebraic expressions. These expressions can be used to find starting cases for continuation methods over a range of parameter values. For many-revolution transfers, if the smaller of either the initial or final orbits is noncircular, chaotic behavior and local minima may be observed using continuation methods to solve for minimum-time orbit transfers by varying parameters or end conditions. This paper examines chaotic discontinuities in the continuation method and offers techniques to help automate parameter searches for minimum-time, continuous-thrust orbit transfers.

EQUATIONS OF MOTION & OPTIMAL CONTROL THEORY

The two-dimensional equations of motion of a spacecraft under continuous thrust may be expressed in polar coordinates which include r as the scalar distance from the attracting center, u as the time rate of change of r, and v as the velocity component perpendicular to u directed along the spacecraft horizon. The polar thrust angle, ϕ, is measured clockwise ("up") from the spacecraft local horizontal. The two-dimensional numerical examples presented in this paper are adequate to explore chaotic behavior and local versus global minima. In general, the minimum-time solution of a two-dimensional problem can be used as a starting point for three-dimensional problems as necessary. The magnitude of the thrust acceleration is $A = T/(m_0 + \dot{m}t)$ where T is the constant thrust magnitude of the propulsion system, and t is the time. If $\dot{m} = 0$, then A is equal to a constant. The only external force we will consider is that of gravity from a single point source. This results in the following differential equations:

$$\dot{r} = u \tag{1}$$

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + A\sin\phi \tag{2}$$

$$\dot{v} = -\frac{uv}{r} + A\cos\phi \tag{3}$$

First, define a scalar function $\mathcal{H}$, the Hamiltonian:

$$\mathcal{H} = \mathcal{L} + \vec{\lambda}^T \vec{f} \tag{4}$$

Using the polar equations of motion in two dimensions, the Hamiltonian is as follows:

$$\mathcal{H} = 1 + \lambda_r u + \lambda_u \left(\frac{v^2}{r} - \frac{\mu}{r^2} + A\sin\phi \right) + \lambda_v \left(-\frac{uv}{r} + A\cos\phi \right) \tag{5}$$

For the Hamiltonian to be stationary with respect to the control variable and the states, we have:

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \tag{6}$$

$$\frac{\partial \mathcal{H}}{\partial \vec{x}} = -\dot{\vec{\lambda}} \tag{7}$$

Equation 6 is the optimality condition, which states that the variation of the cost with respect to the control should be zero on the optimal path. Equation 7 provides a set of first order differential equations that govern the behavior of the Lagrange multipliers. These are the costate equations, which may be integrated along with the state equations through the time interval.

The optimality condition yields the following:

$$\frac{\partial \mathcal{H}}{\partial \phi} = A\left(\lambda_u \cos\phi - \lambda_v \sin\phi\right) = 0 \tag{8}$$

Solving for the polar thrust angle ϕ leads to:

$$\phi = \tan^{-1}\left(\frac{\lambda_u}{\lambda_v}\right) \tag{9}$$

Using the control law given above, we obtain the following:

$$\cos\phi = \frac{\lambda_v}{\sqrt{\lambda_u^2 + \lambda_v^2}} \tag{10}$$

$$\sin\phi = \frac{\lambda_u}{\sqrt{\lambda_u^2 + \lambda_v^2}} \tag{11}$$

To find the costates, use Equation 7:

$$\dot{\lambda}_r = -\frac{\partial \mathcal{H}}{\partial r} = -\lambda_u \left(-\frac{v^2}{r^2} + \frac{2\mu}{r^3} \right) - \lambda_v \frac{uv}{r^2} \tag{12}$$

$$\dot{\lambda}_u = -\frac{\partial \mathcal{H}}{\partial u} = -\lambda_r + \lambda_v \frac{v}{r} \tag{13}$$

$$\dot{\lambda}_v = -\frac{\partial \mathcal{H}}{\partial v} = -\lambda_u \frac{2v}{r} + \lambda_v \frac{u}{r} \tag{14}$$

which completes the set of optimal control equations in the polar case. This result may be used to determine the optimal control law for a spacecraft under continuous thrust. However, there is a significant difficulty inherent in this formulation. Although the costate equations may be derived as shown, they must be initialized to begin a numerical integration procedure such as the shooting method to achieve convergence. We may determine the desired initial and final conditions for the physical states, but there is no guaranteed way to determine the correct, optimal boundary conditions for the costates. If they are too far away from the correct values, the shooting method will fail, but presumably there will be a non-zero radius of convergence for the search technique.

NUMERICAL EXAMPLES

The first example examined in this paper is a minimum-time, continuous thrust magnitude maneuver from a LEO-to-GEO elliptical transfer orbit to circular GEO in the same plane. In order to find the solution to this example, a continuation method was used to start from a case with less than one revolution that converged quickly using the algebraic expressions for the initial values[2]. Then, the magnitude of constant thrust was slowly reduced for additional cases while using the previous values of initial costates and flight time, until the desired level was reached. Canonical units for this example were based on a geosynchronous orbit, where the distance unit was the orbit radius and the period of the circular orbit was 2π time units. The mass unit was based on the initial mass of the spacecraft. In these units, the initial acceleration value was 0.0006081 DU/TU^2, and the specific mass flow rate was -0.00005 MU/TU.

In Fig.1, the initial, elliptical starting orbit is shown starting near the origin and the final orbit is shown with a solid dark circle of unit radius. It is interesting to observe that the optimal control law drives the spacecraft well past the final value of the circular orbit radius for many orbital revolutions on the way to reaching the final circular orbit.

In Fig.2, it may be seen that the semi-major axis grows at first, then at about the halfway point, the eccentricity begins to be reduced to zero to meet the circular final conditions. At no point does the semi-major axis of the intermediate trajectory exceed unity.

The costate history appears as shown in Fig.3. Notice the complicated behavior near the beginning of the trajectory. This phenomenon will be examined further in later examples.

The number of revolutions changes in discrete steps as the thrust magnitude decreases through the continuation parameter search as seen in Fig.4. This is because an integer number of revolutions puts the spacecraft beyond the final radius value for most of the transfer. In order to match the final desired value of orbit radius, this needs to be accomplished somewhere away from the vicinity of an integer number of revolutions. This chaotic, discontinuous behavior apparently arises due to the non-circular starting conditions.

The locus of optimal intial costate values shown in Fig.5 forms a series of segments, rather than a decreasing spiral as in the circle-to-circle case[2]. Once this pattern was observed, it was possible to create additional logic in the continuation method to restart any given search from a region that was more likely to result in convergence by defining three "zones" at the crossing and end points of the segments of the locus.

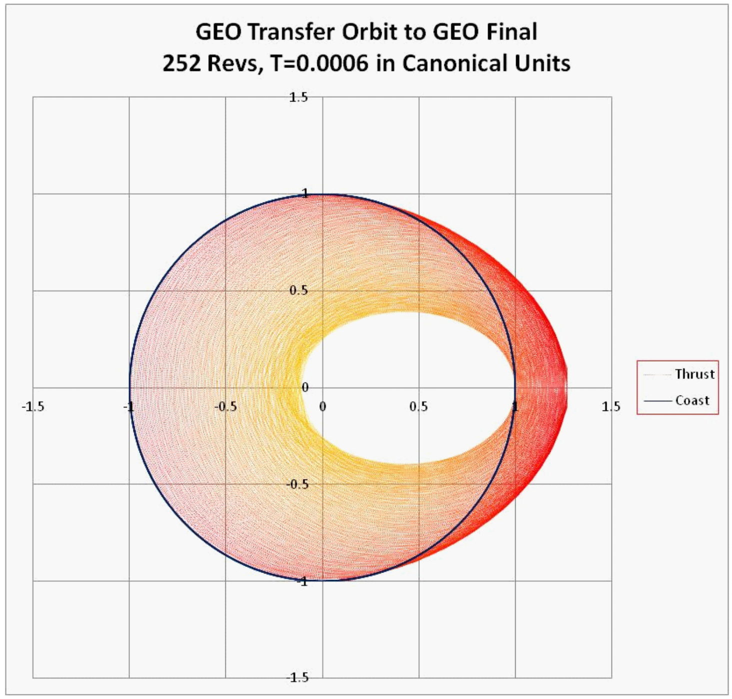

Figure 1: Elliptical starting orbit to tangent circle

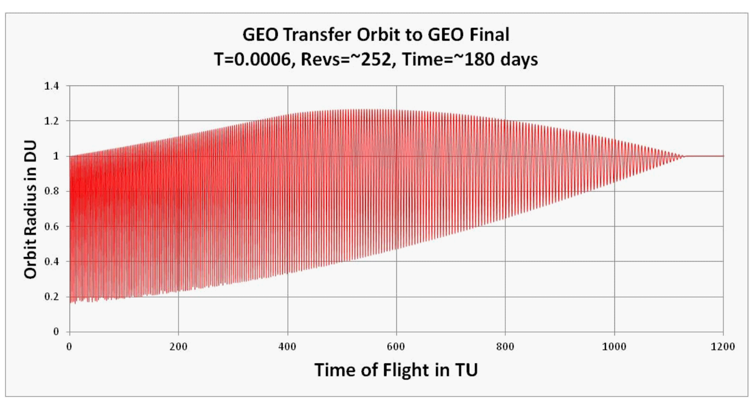

Figure 2: Growth of semi-major axis and circularization

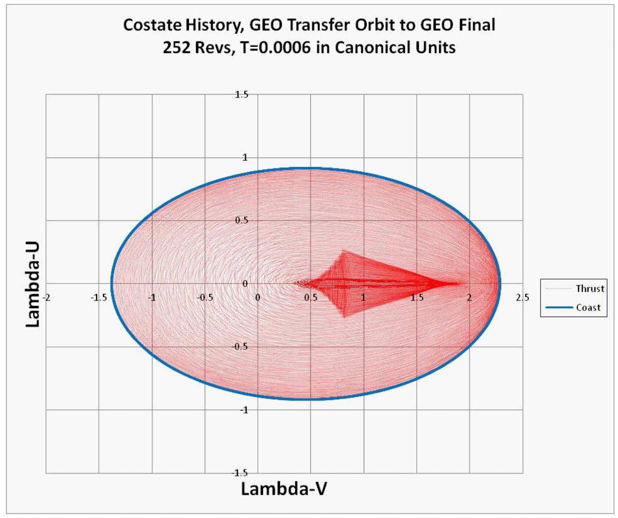

Figure 3: Costate history for ellipse to tangent circle example

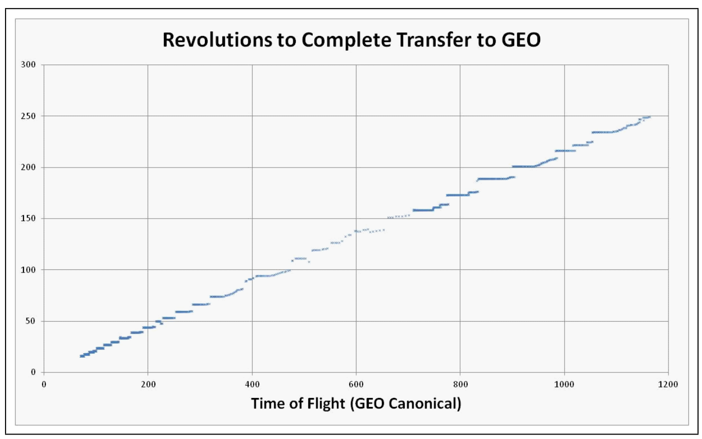

Figure 4: Orbit revolutions vs. time for example case

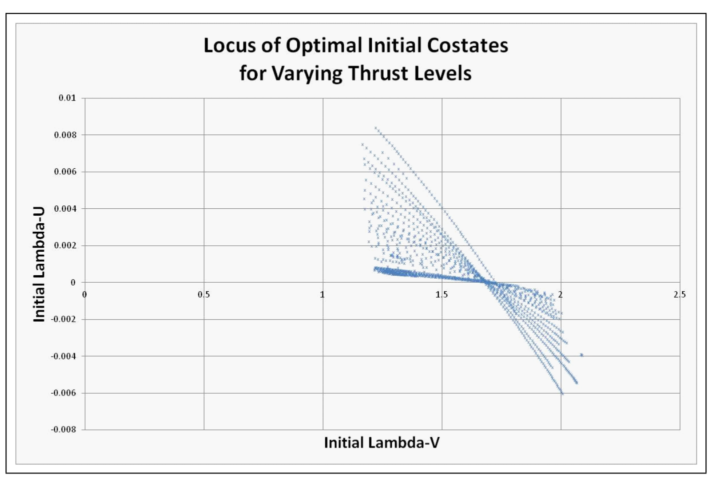

Figure 5: Locus of intial costates for example case

The approximate relationship between flight time and thrust magnitude has the same form as for a circle-to-circle case with $tf = Vacc/T$, where $Vacc = \int_0^{t_f} \left(\frac{T}{m_0 + \dot{m}t} \right) dt$. However, the value of $Vacc$ had to be found numerically from a high thrust case. The numerical value for $Vacc = 0.70651$ was found by taking the product of the flight time and the thrust magnitude in canonical units for an initial case with a relatively high value of 0.01. Once found, the agreement was quite good based on fitting through a single point, as seen in Fig.6. The two curves are nearly indistinguishable on the

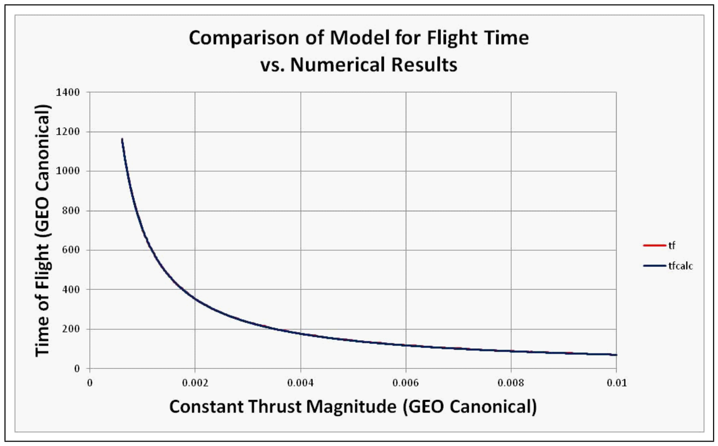

Figure 6: Flight time vs. thrust magnitude for example case

graph, which would indicate that the intemediate solutions during a parameter search remain close to the optimal, minimum time solutions. This relationship can be used to guide the search through continuation methods, by using the $Vacc$ model for flight time as an alternative initial input for new cases as the thrust magnitude parameter is gradually reduced towards the final design value.

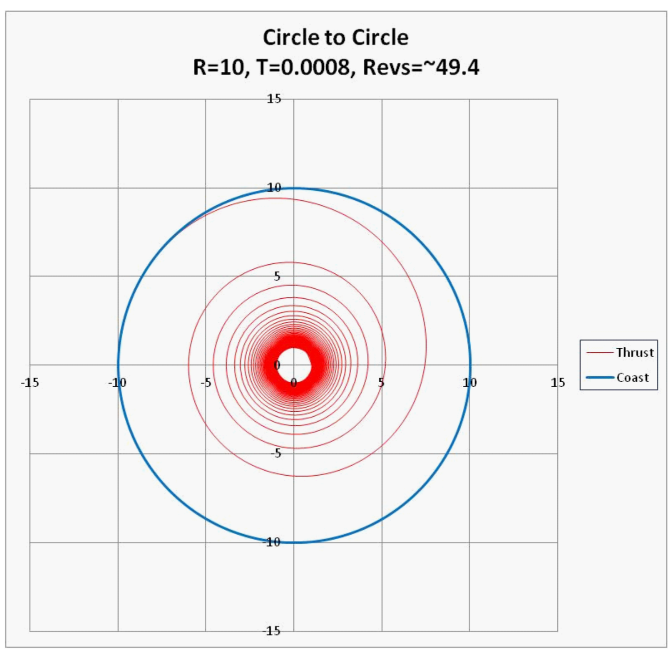

Figure 7: Circle-to-circle reference transfer, R=10

In a second numerical example, a direct comparison is drawn between a transfer from a unit circle to a final circular radius of ten canonical units and a similar case starting with an elliptical starting orbit. The thrust level is the same in both cases, and the final circular orbit is the same size. Fig.7 shows the circle-to-circle reference case with nearly fifty revolutions for the minimum-time transfer.

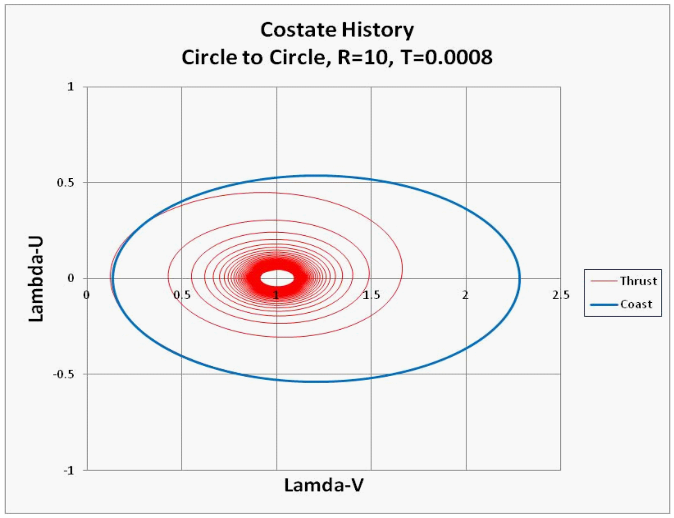

Figure 8: Costate history for reference transfer, R=10

The costate history for the two Lagrange multipliers that are used for the control law, λ_u and λ_v, are plotted in Fig.8 to show their history during the minimum-time transfer. A line drawn from the origin in Fig.8 to a point on the curve shows the thrust angle at any instant as measured counterclockwise from the horizontal axis.

By comparison, Fig.9 shows a similar case where the initial orbit is elliptical with a periapse of unity and an apoapse of three canonical units, for an eccentricity of 0.5. The number of revolutions needed to complete the transfer is much less than the circle-to-circle case, at about 13. However, the physical appearance of the minimum-time trajectory is not much different from the reference case.

On the other hand, from an elliptical starting orbit to final circle, the costate history is very different than spiral behavior of the reference case. The complicated behavior of the costates as a function of time may be seen clearly in Fig.10. Based on this time history, the behavior of the costates during the thrust phase would seem to be quite sensitive to the initial physical conditions. However, after thrust termination during the coast phase, the costate trace is the same elliptical shape as before, except for scaling. The Lagrange multiplier costates may be scaled arbitrarily with no loss in generality because of their form in the variational Hamiltonian. In the case of a transfer from an ellipse to a circular orbit, it was observed that there were discontinuities during the parameter search over thrust magnitude. Further, local minima appeared which could be explored by approaching solutions from different directions in the search space for the total flight time or over the initial values of the Lagrange multipliers. An attempt was made to address the discontinuity and local minima issues by searching backwards in time from a larger starting circle to a smaller elliptical final orbit, but the same behaviors were observed. This should be expected for a reversible process.

As a third numerical example, it was desired to see if the same discontinuities search and local minima would appear over the thrust parameter if the transfer started out on a smaller circular orbit and ended on a non-circular condition. In this case, the transfer starts on a unit circle and ends at a radius of four in canonical units, but with a parabolic escape velocity of $V_{esc} = \sqrt{2}/2$. As seen in Fig.11, the transfer spiral starts in a very similar way to the circle-to-circle case. After about

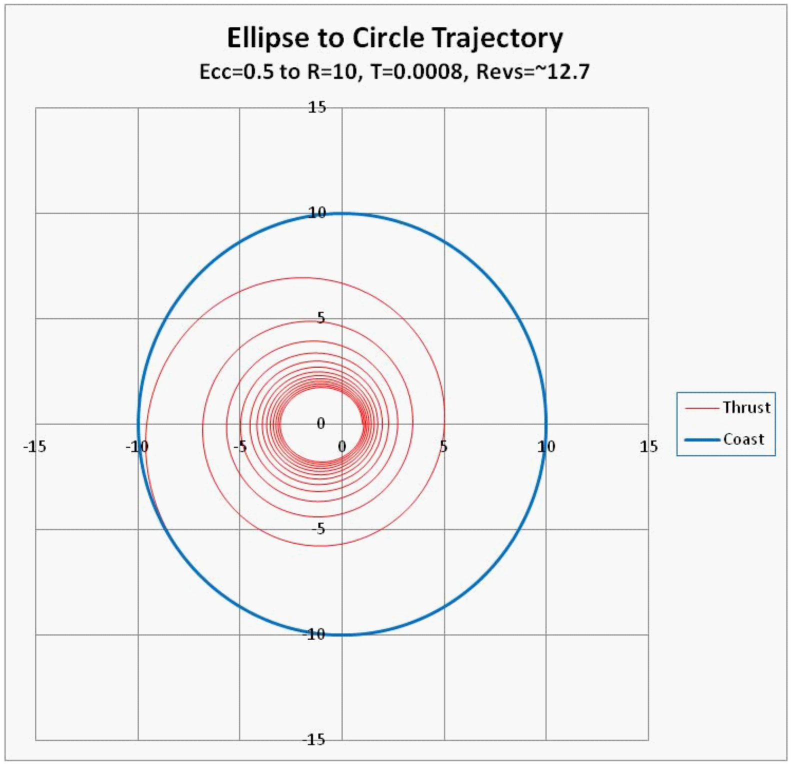

Figure 9: Ellipse to circle trajectory, R=10

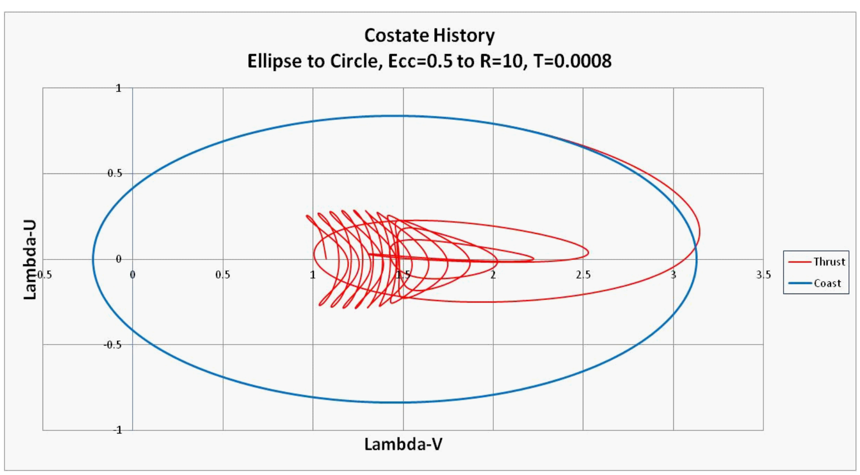

Figure 10: Costate history for ellipse to circle, R=10

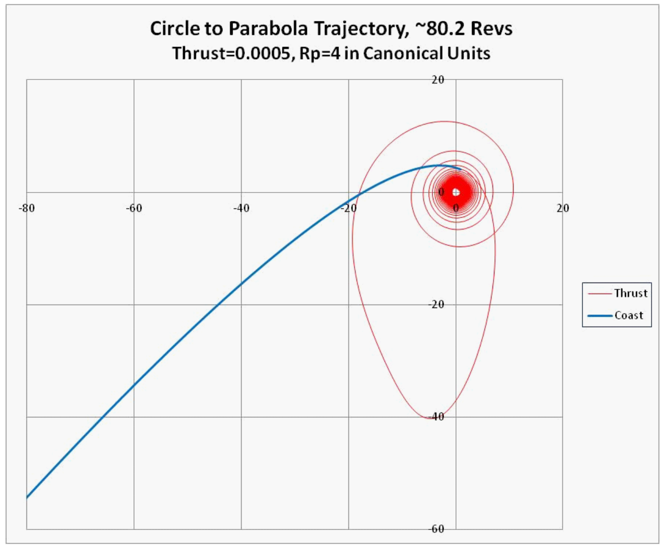

Figure 11: Circle to parabolic trajectory, R=4

80 revolutions, however, the radius begins to decrease to achieve a flyby at the desired final radius. Based on the braking maneuver seen in Fig.11 prior to the flyby point, this behavior is reminiscent of impulsive bi-elliptic transfers which take advantage of smaller delta-v requirements at larger distances for greater overall efficiency. Based on numerical investigation, no discontinuities were found in the thrust parameter search, and no local minima were found outside of the converged solution for this example. It would appear that if the example resembles a low-thrust outward spiral from an initial circular orbit, there is no obvious dependence on the total transfer angle, and apparently the discontinuities and local minima can be avoided.

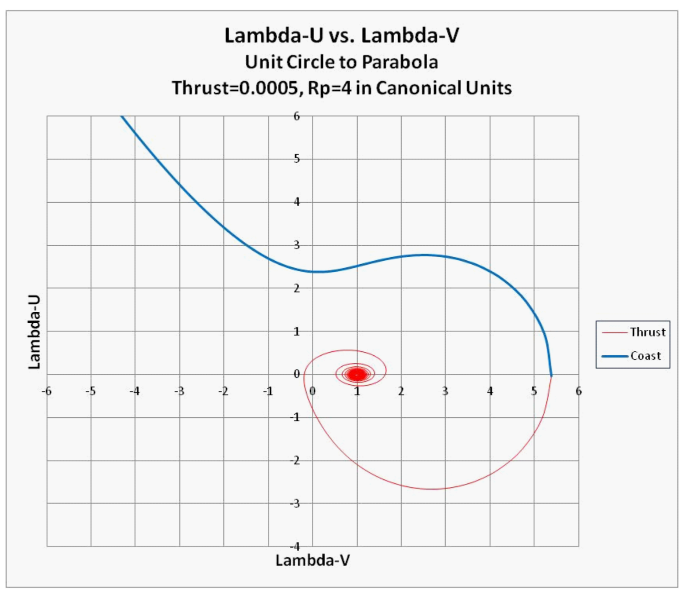

Figure 12: Costate history for circle to parabola example, R=4

For the circle-to-parabola example, the costate history shown in Fig.12 starts off looking very similar to the circle-to-circle case with no complicated path crossings as in the ellipse-to-circle case.

At the end, the costate magnitude increases indefinitely in the same way as the physical radius of the orbit itself. Based on this series of numerical examples, it would appear that the minimum-time solution is a global minimum over the search space as long as the costate history path does not cross itself for a multiple-revolution orbit transfer, even if the physical path does cross itself.

CONCLUSIONS

Since the relationship between accumulated velocity change and the magnitude of the constant thrust is only dependent on the time of flight, one may use a single sample point to establish the relationship with very good agreement for a large range of thrust values with the same boundary conditions, even for examples with discontinuities in the thrust parameter search method and the presence of local minima. This provides a means to check the optimality of converged results of any particular case during the implementation of the continuation method. Regarding global versus local minima, it would appear that the minimum-time solution is a global minimum over the search space as long as the costate history path does not cross itself for a multiple-revolution orbit transfer, even if the physical path does cross itself.

References

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[3] Bryson, A. E. Jr., and Ho, Y., *Applied Optimal Control*, Ginn and Company, Waltham, Massachusetts, 66-69, 1969.