FALSE-OBJECT IDENTIFICATION FOR SPACE SURVEILLANCE CATALOG MAINTENANCE

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ABSTRACT

A space object from a surveillance catalog of space objects that is predicted to be in the field of view of a tracking sensor may not be detected by a tracking sensor because of viewing conditions, or because either the estimated orbit of the space object has large error or the catalog object does not actually exist. In these two latter cases we call such a catalog object an invalid object (a false or lost track). Identification of invalid catalog objects is an essential function for maintenance of a space surveillance catalog.

An invalid catalog object is not likely to be associable with any measurements (observations) in a sequence of data collects from tracking sensors. The sequential probability of validity over multiple frames of data is cumulative evidence of whether the catalog object is valid or invalid. The catalog object is deemed to be invalid when the sequential probability of validity is sufficiently close to zero.

The single-frame and sequential probabilities of validity of a catalog object are determined by first computing the maximum likelihood association of the tracks (the estimated orbits) and observations, and then by updating a sequential likelihood ratio test for each track. The probability that each track is valid is computed from likelihood ratios. The algorithm is simpler and computationally faster than the general Multiple Hypothesis Testing (MHT) algorithm that associates multiple measurements and multiple objects over multiple frames of data. The sequential data collects do not have to correspond to the same set of catalog objects, and the data collects can be separated in time and can come from tracking sensors that are geographically separated, have different views, and that report different types of measurements (angles, range, range rate, position vectors, or position and velocity vectors). The measurement data can include feature (attribute) data. The only condition is that the catalog object under consideration is predicted to be in or near the field of view of the tracking sensor in multiple data collects; whether or not there are any associable measurements in the data collects is determined by the algorithm.

1. INTRODUCTION

A space surveillance catalog of orbiting objects is maintained by processing surveillance data. Estimates of the orbits[1] (called tracks) of existing catalog objects are updated using surveillance data, and new or tentative catalog objects may be spawned from data that are not associated with established catalog objects. A catalog object may be initiated by a tracking system, or an operator may create an "analyst object" or "potential object" from unassociated observations that appear to fit an orbit. Observation data might not be reported for certain catalog objects predicted to be in the field of view of the surveillance sensor, which can happen if the object is not detected, if the orbit prediction error is large, if the catalog object is a false track, or if the track is lost due to a maneuver of the space object. When observation data is not reported for a catalog object, the object is referred to as "missing". An essential catalog maintenance function is to identify and remove false and lost tracks (objects) from the catalog. A false or lost track will be called an invalid track.

[1]Estimates of the orbits of space objects are equivalent to target tracks in the target tracking literature. In space surveillance, a "track" sometimes refers to a sequence of position measurements of the space object. We shall refer to the estimated orbit as a track.

In this work, we develop a computationally fast algorithm to validate or invalidate the tracks (estimated orbits) in a catalog of space objects so that invalid tracks can be eliminated from the catalog. The algorithm computes the sequential probability, based on surveillance data, that a track is valid. For each frame of data, the algorithm computes the maximum likelihood association and non-association of the observations and tracks, and then updates a sequential likelihood ratio for each track. The probability that a track is valid is computed from its likelihood ratio. The sequential probability is cumulative evidence of whether the track is valid or invalid. The track (catalog object) is deleted from the catalog when the probability of being valid is sufficiently close to zero, and it is a valid (confirmed) track when the probability is sufficiently close to one.

1.1. Background

Radar and optical surveillance sensors return a set of measurements of the location of orbiting objects, typically in a sequence of frames or scans of data. Each frame of data may include spurious measurements and measurements of non-catalog objects. The sensors do not report which measurements belong to which tracks. Furthermore, the measurements and target state estimates contain error and so the correct association of tracks and measurements may not be obvious or unambiguous. A statistical data association algorithm is therefore needed. Original applications, and indeed motivations, for the development of statistical data association methods include air, surface, and subsurface target tracking [1–13]. Application of data association methods to space object tracking for space surveillance and space situational awareness is relatively more recent [14–22]. A brief overview of data association methods is provided here.

Suppose we have N tracks of space objects in our catalog and M observations (measurements) in a single-frame of data. The tracks are predicted to the time of the observations, and those that are expected to be within the field-of-view (FOV) or field-of-regard (FOR) of the tracking sensor with high probability are selected as candidates for matching with the observations. We form a score matrix (also called a cost matrix) S of association scores $s_{j,i}$,

$$S = \begin{array}{c} \\ T_1 \\ T_2 \\ \vdots \\ T_N \end{array} \begin{array}{c} \begin{array}{cccc} O_1 & O_2 & \cdots & O_M \end{array} \\ \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1M} \\ s_{21} & s_{22} & \cdots & s_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ s_{N1} & s_{N2} & \cdots & s_{NM} \end{bmatrix} \end{array}$$

where the $s_{j,i}$ measure in some sense the closeness of track T_j to observation O_i. Various types of scores are cited in the next paragraph. We seek the best set of unique (one-to-one) matches $\{j, \boldsymbol{m}(j)\}$, $1 \leq j \leq N$, where $\boldsymbol{m}(j)$ is the index of the match for track T_j. The algorithm used to find the best match depends on how "best" is defined. A Nearest Neighbor (NN) matching algorithm, also called a Greedy Selection algorithm, first finds the match with the lowest score, then the lowest of the remaining scores after eliminating the row and column corresponding to the first match, et cetera. (The highest score value may be sought, depending on how the score is defined.) The NN algorithm is not optimal in any sense and does not necessarily find a unique (one-to-one) set of matches. The Global Nearest Neighbor (GNN) solution is defined as the minimum sum of matching scores, $\min_{\boldsymbol{m}} \sum_{j=1}^{N} s_{j,\boldsymbol{m}(j)}$, where $\boldsymbol{m}$ is a unique set of matches. However, the GNN solution is suboptimal because the score matrix as just defined does not account for spurious measurements, new tracks, and invalid tracks, and the choice of scores is arbitrary.

The scores could be the Mahalanobis distances between the tracks and measurements [8] or modified Mahalanobis distances [4, §4.3.1] and [5, §6.4]. Information theoretic measures such as the Bhattacharyya distance [18, 21, 22], Kullback-Liebler divergence [18–20], and entropy [18] have also been considered for the scores, although the the association algorithm (e.g., NN or GNN) is not specified in these works. For any type of score, the NN approach is clearly suboptimal. The meaning of the sum of information measures in the GNN solution is unclear for multiple tracks and measurements, since joint probability densities should be used in the information measures. Optimal scores are the log-likelihood of joint density functions, or log-likelihood ratios, of the tracks and measurements, presented later in this work.

The original work on multiple hypothesis tracking[2] (MHT) [1–3] approached the problem of measurement-to-track association for multiple frames as a multiple hypothesis problem where the hypotheses grow rapidly in number with each frame of data until association decisions are made and the hypothesis tree is pruned. Two approaches to implementing MHT are hypothesis-oriented and track-oriented multiple hypothesis testing (HOMHT and TOMHT) [5–7, 16, 23]. The TOMHT is easier to implement and comprehend than the HOMHT. Inherent in the MHT formulation are track initiation, track maintenance, and track termination. In the MHT framework, a number of frames of data are used in deciding the measurement-to-track association. The association decision is delayed until a fixed number of frames of data are collected, at which point the decision tree is pruned to prevent the number of association hypotheses from growing too large. In addition, a decision is made to initiate new tracks based on non-associated measurements and to terminate (delete) tracks that have been propagated too long without being updated. Multi-dimensional association algorithms have been developed to efficiently compute the associations over multiple tracks and multiple frames of data [24–27]. Despite advances in numerically efficiency and computational speed, the HOMHT and TOMHT algorithms are too computationally intensive for catalog maintenance, at least for our present purpose.

The Probabilistic Data Association (PDA) algorithm [4, 5, 8–10] is effective in associating one of several measurements with a given target, but does not work well for closely-spaced objects (relative to the error in the tracks and measurements). The Joint Probabilistic Data Association (JPDA) algorithm [4, 5, 7–10] is used maximize the joint probability of the actual and false measurements and the actual and false tracks. The PDA and single-frame JPDA algorithms require are simpler and require much less computation than the HOMHT and TOMHT algorithms. The PDA and JPDA can be applied in a Nearest Neighbor mode, which selects the most probable measurement for each track, or in an All Neighbors mode, which averages the measurements weighted by their probability [10, §6.2.1]. In both the PDA and JPDA, a gating algorithm is used to cull tracks and measurements that have a low probability of association.

The track validation (aka false-object identification) algorithm developed in this work is similar to the track deletion algorithm in [5, §6.2, Ch. 9] and [6], but we take a different approach to formulating the problem. In particular, we maximize the joint association probability rather than a joint density function of the data (although the result is similar), and we show explicitly the link between maximizing the association probability and minimizing the trace of a score matrix. This should help clarify the JPDA algorithm.

The track validation algorithm can incorporate attribute data (also called feature variables) [5, Ch. 8], [10, §6.5, §6.6], [11, §3.4.7], [12]. Examples of attribute data are visual magnitude of the observed object as measured by an optical sensor and predicted by viewing conditions and target orientation, signal strength, Doppler, and spin rate signature sensed by a radar. Attribute data can potentially improve the track validation performance by distinguishing objects that are closely spaced in position, particularly angular position.

1.2. Organization

The false catalog object identification (or track deletion) algorithm is derived in the next section. We first define mathematical models for the state vector and measurement process in Section 2.1. Spurious measurement and invalid track densities and detection event probabilities are specified in Section 2.2 for the four combinations of detection, non-detection, valid track, and invalid track events. The results of

[2]The term Multiple Hypothesis Testing (MHT) is a general subject area in statistics. In the literature on target tracking, MHT means "multiple hypothesis tracking" and refers to a specific set of hypotheses to associate measurements and tracks. The reader should not confuse these definitions.

Section 2.2 are combined to obtain the density functions in Section 2.3 for the four events. The joint association probability is given in Section 2.4, from which the Maximum *a Posteriori* (MAP) and Maximum Likelihood (ML) estimates of the track–measurement association are derived. In Section 2.5 a score matrix is defined such that the MAP or ML optimal association can be computed. Next, in Section 2.6, a likelihood ratio test for space object track validation/invalidation is derived. Practical bounds on parameters that determine the association thresholds and track validity are derived in Section 2.7. Expressions for the minimum number of frames of data to decide whether a track is valid or invalid are derived in Section 2.8. The derivations in Sections 2.5 and 2.6 are similar to derivations in [2, 5], but some notable differences are noted in those sections. Methods to detect and handle possible assignment ambiguity are given in Section 2.9. Some simulation results are given in Section 3 and the conclusion is provided in Section 4.

2. ALGORITHM FOR CATALOG OBJECT TRACK VALIDATION

2.1. State and Measurement Model

The state of a space object (track) T_j, with index j, is given by the state vector $\boldsymbol{x}_j$. We shall omit a time index until needed for clarity. For our purpose, the state vector comprises position and velocity in Earth-Centered Inertial (ECI) J2000.0, though other descriptions such as Keplerian or equinoctial elements could be used. The state equation (derived from orbital dynamics) does not have to be specified for the purpose of data association, and so are omitted here.

A d-dimensional measurement vector z_i, indexed by i, is obtained through a (generally) nonlinear observation model $\boldsymbol{h}_i(\boldsymbol{x}_j)$ and corrupted by zero-mean white Gaussian noise $\boldsymbol{v}_i$,

$$z_i = \boldsymbol{h}_i(\boldsymbol{x}_j) + \boldsymbol{v}_i \tag{1}$$

The $d \times d$ covariance matrix of the measurement error $\boldsymbol{v}_i$ is R_i. The predicted measurement is

$$\hat{z}_j = \boldsymbol{h}_j(\hat{\boldsymbol{x}}_j) \tag{2}$$

The measurement residual is given by

$$\tilde{z}_{j,i} = z_i - \hat{z}_j \tag{3}$$

The estimated state $\hat{\boldsymbol{x}}_j$ of object T_j has error covariance S_j, so the covariance matrix of the residual is

$$\Sigma_{j,i} = H_i S_j H_i^T + R_i \tag{4}$$

where the linearized observation matrix H_i is

$$H_i = \left.\frac{\partial \boldsymbol{h}_i}{\partial \boldsymbol{x}^T}\right|_{\boldsymbol{x} = \hat{\boldsymbol{x}}_j} \tag*{*(5)}$$

Assuming the track (the estimate of the orbit) has not already been updated with the measurement, the probability density function for the residual is the Normal (Gaussian) density function

$$\begin{aligned} p(\tilde{z}_{j,i}) &= \mathcal{N}(\tilde{z}_{j,i}; \boldsymbol{0}, \Sigma_{j,i}) \\ &= \frac{1}{(2\pi)^{d/2}|\Sigma_{j,i}|^{1/2}} \exp[-\tfrac{1}{2}\, \tilde{z}_{j,i}^T\, \Sigma_{j,i}^{-1}\, \tilde{z}_{j,i}] \end{aligned} \tag{6}$$

The form of the partial derivative in Eq. (5) depends on the definition of the estimated state vector $\hat{\boldsymbol{x}}_j$, the type of measurements (angles, range, etc.) and the position, inertial velocity, and orientation of the tracking sensor. Ideally the measurement z_i and its covariance R_i should be in the coordinate system of the tracking sensor, and the location, inertial velocity, and orientation of the tracking sensor should be known. Analytical expressions for $\boldsymbol{h}_i$ and H_i depend on the particular sensors and are beyond the scope of this article, and so are omitted here. See [28, 29] for detailed discussions and derivations of observation models.

The tracking sensor reports M measurements z_i, $1 \le i \le M$, in a single data collect. We can assume that the measurements are simultaneous, though in reality small adjustments may be needed. Given the sensor orientation and configuration, we find N catalog objects T_j, $1 \le n \le N$, in or near the FOV of the tracking sensor. There can be more or fewer measurements than objects, so it is possible that $M \ne N$. We assume that there is at most one measurement reported for each detected object in a given data collect. The correspondence between the measurement index i and the target index j is not reported by the tracking sensor and so the correspondence must be determined by a data association algorithm.

Suppose we have a set of M vector measurements

$$\boldsymbol{Z} = [z_1\ z_2\ \ldots\ z_M] \tag{7}$$

and a set of N predicted vector measurements computed from the estimated (catalog) state vectors of objects

$$\hat{\boldsymbol{Z}} = [\hat{z}_1\ \hat{z}_2\ \ldots\ \hat{z}_N] \tag{8}$$

The set of measurements is obtained from one of a sequence of measurement frames (or scans).[3] The frame index k is omitted until needed. The catalog objects are those that are predicted, with sufficient probability, to be in the field of view of the tracking sensor at the time of the measurements. The list of catalog objects may change from one frame to another as objects enter and leave the field of view of the tracking sensor. For the purpose of exposition of our derivations, we will not vary the objects in the list, although the predicted measurements will change in each frame.

A tracking sensor might not detect all space objects in its FOV or FOR, and the frame of measurement data may include spurious measurements (false detections, a.k.a. false alarms) and measurements of non-catalog objects (new, non-tracked, and lost objects). A catalog object, represented by its track (estimated orbit) may also be invalid (a false or lost track). The occurrence of spurious measurements and invalid catalog objects in the FOV or FOR of a tracking sensor can be modeled as Poisson processes (a "parametric" model) or as random events from a diffuse density function (a "non-parametric" model) [7, 8, 11]. We will use density functions in our derivations; the parametric model is not considered in this work.

A "validation gate" is often used in data association to reduce the number of measurements and tracks that have to be considered for association[5, §6.3], [4, §4.2], [8], [9, §2.2.2.2]. These include rectangular and ellipsoidal gates. The ellipsoidal gate is a comparison of the Mahalanobis distance (between a track and a measurement) to a threshold. The threshold is determined either by a χ^2 probability or a maximum likelihood criterion. The maximum likelihood criterion in [4, §4.2.2] and [5, §6.3.2] can be derived from the inequality in Eq. (43) below. The maximum likelihood criterion in [9, §2.2.2.2, Eq. (2.7)] can be derived from Eq. (45) below. We will not focus on gating here because it requires substantial computation, there will not be a large number of measurements and targets in the FOV or FOR of our tracking sensor, and because gating would essentially duplicate the work of the data association and track validation algorithms.

2.2. Probability Densities for Measurements and Tracks Under Hypotheses

The event that a detection occurred is denoted $\mathcal{D}_1$ and the event that no detection occurred is denoted $\mathcal{D}_0$. Let $\mathcal{D}$ be a detection event such that $\mathcal{D} = \mathcal{D}_1$ or $\mathcal{D} = \mathcal{D}_0$. The event that a track is valid is denoted $\mathcal{H}_1$ and

[3] We assume that the measurements are all of the same type so that their units are consistent, although for mixed types we can non-dimensionalize the density functions by multiplying by their respective surveillance volumes or by using likelihood ratios.

the event that a track is invalid is denoted $\mathcal{H}_1$. Let $\mathcal{H}$ be the hypothesis that $\mathcal{H} = \mathcal{H}_1$ or $\mathcal{H} = \mathcal{H}_0$. We will derive expressions for the probability density of the track state, conditioned on $\mathcal{D}$ and $\mathcal{H}$. The probability density of the track state will be a function of a measurement tentatively associated with the track.

We will define test statistics $\boldsymbol{y}_{j,i}$ that depend on the predicted measurements $\hat{z}_j$ and measurements z_i under the four combinations of hypotheses of detection, non-detection, valid track, and invalid track. For the moment we drop the subscripts j and i. From the Chain Rule of Probability, the joint density of the test statistic $\boldsymbol{y}$, detection event $\mathcal{D}$, and track validity hypothesis $\mathcal{H}$ is given by

$$p(\boldsymbol{y}, \mathcal{D}, \mathcal{H}) = p(\boldsymbol{y} \,|\, \mathcal{D}, \mathcal{H})\, P(\mathcal{D} \,|\, \mathcal{H})\, P(\mathcal{H}) \tag{9}$$

From the definition of conditional probability, we have

$$p(\boldsymbol{y}, \mathcal{D} \,|\, \mathcal{H}) = \frac{p(\boldsymbol{y}, \mathcal{D}, \mathcal{H})}{P(\mathcal{H})} \tag{10}$$

Thus the joint density of the track state and detection event, conditioned on the track validity hypothesis is

$$p(\boldsymbol{y}, \mathcal{D} \,|\, \mathcal{H}) = p(\boldsymbol{y} \,|\, \mathcal{D}, \mathcal{H})\, P(\mathcal{D} \,|\, \mathcal{H}) \tag{11}$$

The density function $p(\boldsymbol{y} \,|\, \mathcal{D}, \mathcal{H})$ in Eq. (11) depends on detection or non-detection of a tracked space object and the validity or invalidity of the track.

Under the four hypotheses $\{\mathcal{D}, \mathcal{H}\}$, the test statistics $\boldsymbol{y}_{n,m}$ and their probability density functions are[4]

$$\begin{aligned}
&\{\mathcal{D}_1, \mathcal{H}_1\}: && \boldsymbol{y}_{j,i} &&= z_i - \hat{z}_j && p(\boldsymbol{y}_{j,i} &&|\, \mathcal{D}_1, \mathcal{H}_1) = \mathcal{N}(\boldsymbol{y}_{j,i}; \boldsymbol{0}, \Sigma_{j,i}) && \text{(12a)}\\
&\{\mathcal{D}_0, \mathcal{H}_1\}: && \boldsymbol{y}_{j,\ell+M} &&= \hat{z}_j && p(\boldsymbol{y}_{j,\ell+M} &&|\, \mathcal{D}_0, \mathcal{H}_1) = \lambda_t && \text{(12b)}\\
&\{\mathcal{D}_1, \mathcal{H}_0\}: && \boldsymbol{y}_{j+N,i} &&= z_i && p(\boldsymbol{y}_{j+N,i} &&|\, \mathcal{D}_1, \mathcal{H}_0) = \lambda_s && \text{(12c)}\\
&\{\mathcal{D}_0, \mathcal{H}_0\}: && \boldsymbol{y}_{j+N,\ell+M} &&= \hat{z}_j && p(\boldsymbol{y}_{j+N,\ell+M} &&|\, \mathcal{D}_0, \mathcal{H}_0) = \lambda_u && \text{(12d)}
\end{aligned}$$

for $1 \le i \le M$, $1 \le j \le N$, and $1 \le \ell \le N$. Under $\{\mathcal{D}_1, \mathcal{H}_1\}$, a track and measurement are associated and the test statistic is the residual error, which has a Normal density. Under $\{\mathcal{D}_0, \mathcal{H}_1\}$, there is no measurement to associate with the valid track, and the test statistic (the predicted measurement from the track) is uniformly distributed over the FOV or FOR with density λ_t. Under $\{\mathcal{D}_1, \mathcal{H}_0\}$, there is no valid track to associate with the measurement, and the test statistic (the measurement) is uniformly distributed over the FOV or FOR with density λ_t. Under $\{\mathcal{D}_0, \mathcal{H}_0\}$, the invalid track is uniformly distributed over the FOV or FOR with density λ_u. Ultimately λ_s, λ_t, and λ_u are design values chosen to establish decision thresholds and rate of convergence of the probability of validity. Practical formulas for λ_s, λ_t, and λ_u are discussed in Sections 2.7 and 2.8.

The probability of detection (subject to gating) of a valid catalog object in the sensor's FOV or FOR is

$$P(\mathcal{D}_1 \,|\, \mathcal{H}_1) = P_{\mathrm{D}} \tag{13a}$$

Since $P(\mathcal{D}_0 \,|\, \mathcal{H}_1) = 1 - P(\mathcal{D}_1 \,|\, \mathcal{H}_1)$, the probability of non-detection when the catalog object is valid

$$P(\mathcal{D}_0 \,|\, \mathcal{H}_1) = 1 - P_{\mathrm{D}} \tag{13b}$$

The probability of detection when the catalog object is invalid is the false-alarm detection probability

$$P(\mathcal{D}_1 \,|\, \mathcal{H}_0) = P_{\mathrm{F}} \tag{13c}$$

Since $P(\mathcal{D}_0 \,|\, \mathcal{H}_0) = 1 - P(\mathcal{D}_1 \,|\, \mathcal{H}_0)$, the probability of non-detection when the catalog object is invalid is

$$P(\mathcal{D}_0 \,|\, \mathcal{H}_0) = 1 - P_{\mathrm{F}} \tag{13d}$$

[4]The null and alternate hypotheses for the tracks and measurements are $\{\mathcal{D}_1, \mathcal{H}_1\}$ and $\{\mathcal{D}_0, \mathcal{H}_1\}$, $\{\mathcal{D}_1, \mathcal{H}_0\}$, and $\{\mathcal{D}_0, \mathcal{H}_0\}$. We do need the subterfuges "pseudo-measurements", "pseudo-tracks", or "dummy-tracks" defined in other works such as [2], [4, §9.5], [5, §9.5.3], [7].

2.3. Probability Density Functions

Using Eq. (11), Eqs. (12), and Eqs. (13), the probability density functions $p_{n,m}$ of the test statistics $\boldsymbol{y}_{n,m}$ under the four hypotheses $\{\mathcal{D}\,|\,\mathcal{H}\}$, for $1 \le i \le M$, $1 \le j \le N$, and $1 \le \ell \le N$, are given by

$$\begin{aligned} p_{j,i}(\boldsymbol{y}_{j,i}) &= p(\boldsymbol{y}_{j,i}, \mathcal{D}_1\,|\,\mathcal{H}_1) \\ &= p(\boldsymbol{y}_{j,i}\,|\,\mathcal{D}_1, \mathcal{H}_1)\, p(\mathcal{D}_1\,|\,\mathcal{H}_1) \\ &= \mathcal{N}(\boldsymbol{y}_{j,i}; \mathbf{0}, \Sigma_{j,i})\, P_\text{D} \\ &= \frac{1}{(2\pi)^{d/2}|\Sigma_{j,i}|^{1/2}} \exp\left[-\tfrac{1}{2}\,\boldsymbol{y}_{j,i}^T\,\Sigma_{j,i}^{-1}\,\boldsymbol{y}_{j,i}\right] P_\text{D} \end{aligned} \tag{14a}$$

$$\begin{aligned} p_{j,\ell+M}(\boldsymbol{y}_{j,\ell+M}) &= p(\boldsymbol{y}_{j,\ell+M}, \mathcal{D}_0\,|\,\mathcal{H}_1) \\ &= p(\boldsymbol{y}_{j,\ell+M}\,|\,\mathcal{D}_0, \mathcal{H}_1)\, p(\mathcal{D}_0\,|\,\mathcal{H}_1) \\ &= \lambda_t(1 - P_\text{D}) \end{aligned} \tag{14b}$$

$$\begin{aligned} p_{j+N,i}(\boldsymbol{y}_{j+N,i}) &= p(\boldsymbol{y}_{j+N,i}, \mathcal{D}_1\,|\,\mathcal{H}_0) \\ &= p(\boldsymbol{y}_{j+N,i}\,|\,\mathcal{D}_1, \mathcal{H}_0)\, p(\mathcal{D}_1\,|\,\mathcal{H}_0) \\ &= \lambda_s P_\text{F} \end{aligned} \tag{14c}$$

$$\begin{aligned} p_{j+N,\ell+M}(\boldsymbol{y}_{j+N,\ell+M}) &= p(\boldsymbol{y}_{j+N,\ell+M}, \mathcal{D}_0\,|\,\mathcal{H}_0) \\ &= p(\boldsymbol{y}_{j+N,\ell+M}\,|\,\mathcal{D}_0, \mathcal{H}_0)\, p(\mathcal{D}_0\,|\,\mathcal{H}_0) \\ &= \lambda_u(1 - P_\text{F}) \end{aligned} \tag{14d}$$

2.4. Joint Association Probability

Consider the likelihood ratios for detection and non-detection of a space object,

$$\frac{p_{j,i}}{p_{j,\ell+M}} = \frac{p(\boldsymbol{y}_{j,i}, \mathcal{D}_1\,|\,\mathcal{H}_1)}{p(\boldsymbol{y}_{j,\ell+M}, \mathcal{D}_0\,|\,\mathcal{H}_1)} \tag{15a}$$

$$\frac{p_{j+N,i}}{p_{j+N,\ell+M}} = \frac{p(\boldsymbol{y}_{j+N,i}, \mathcal{D}_1\,|\,\mathcal{H}_0)}{p(\boldsymbol{y}_{j+N,\ell+M}, \mathcal{D}_0\,|\,\mathcal{H}_0)} \tag{15b}$$

The first is the likelihood ratio for detection when the track is valid and the second is the likelihood ratio for detection when the track is not valid. If the inequality

$$\frac{p_{j,i}}{p_{j,\ell+M}} > \frac{p_{j+N,i}}{p_{j+N,\ell+M}} \tag{16}$$

holds, then we would say that detection under $\mathcal{H}_1$ is more likely than under $\mathcal{H}_0$. Cross-multiply terms in Eq. (16) to get

$$p_{j,i}\, p_{j+N,\ell+M} > p_{j+N,i}\, p_{j,\ell+M} \tag{17}$$

If the inequality holds, we base our data association on the density functions on the left, otherwise we base our data association on the density functions on the right.

Let the permutation vector $\boldsymbol{\varphi}$ define a correspondence between the tracks and measurements under the four detection events $\{D\,|\,\mathcal{H}\}$. Define the set $\boldsymbol{Y} = \{\boldsymbol{Z}, \hat{\boldsymbol{Z}}\}$ of measurements and predicted measurements.

The conditional joint density function of $\boldsymbol{Y}$ under the permutation $\boldsymbol{\varphi}$ is

$$p(\boldsymbol{Y}\,|\,\boldsymbol{\varphi}) = \prod_{j=1}^{N} p_{j,\boldsymbol{\varphi}(j)}(y_{j,\boldsymbol{\varphi}(j)})\, p_{j+N,\boldsymbol{\varphi}(j+N)}(y_{j+N,\boldsymbol{\varphi}(j+N)}) \tag{18}$$

Since an optimal assignment depends on the data, $\boldsymbol{\varphi}$ is a random variable that is to be estimated. From Bayes' Rule, the posterior probability of $\boldsymbol{\varphi}$ given the set of test statistics is

$$P(\boldsymbol{\varphi}\,|\,\boldsymbol{Y}) = \frac{p(\boldsymbol{Y}\,|\,\boldsymbol{\varphi})\, P(\boldsymbol{\varphi})}{p(\boldsymbol{Y})} \tag{19}$$

where $P(\boldsymbol{\varphi})$ is the *a priori* probability of $\boldsymbol{\varphi}$. The posterior probability becomes a likelihood function when evaluated at the test statistics. The maximization of $P(\boldsymbol{\varphi}\,|\,\boldsymbol{Y})$ does not depend on $p(\boldsymbol{Y})$ because $p(\boldsymbol{Y})$ is independent of $\boldsymbol{\varphi}$. The Maximum *a Posteriori* (MAP) probability $P(\boldsymbol{\varphi}^*\,|\,\boldsymbol{Y})$ is achieved by the permutation vector $\boldsymbol{\varphi}^*$ that maximizes $p(\boldsymbol{Y}\,|\,\boldsymbol{\varphi})P(\boldsymbol{\varphi})$. Thus we have the MAP assignment

$$\boldsymbol{\varphi}^* = \boldsymbol{\varphi}^*_{\mathrm{MAP}} = \arg\max_{\boldsymbol{\varphi}} p(\boldsymbol{Y}\,|\,\boldsymbol{\varphi})\, P(\boldsymbol{\varphi}) \tag{20}$$

In the usual circumstance that we have no prior knowledge of $\boldsymbol{\varphi}$, we assume that all permutations are equally likely. The Maximum Likelihood (ML) probability $P(\boldsymbol{\varphi}^*\,|\,\boldsymbol{Y})$ is then achieved by the permutation vector $\boldsymbol{\varphi}^*$ that maximizes $p(\boldsymbol{Y}\,|\,\boldsymbol{\varphi})$. Thus we have the ML assignment

$$\boldsymbol{\varphi}^* = \boldsymbol{\varphi}^*_{\mathrm{ML}} = \arg\max_{\boldsymbol{\varphi}} p(\boldsymbol{Y}\,|\,\boldsymbol{\varphi}) \tag{21}$$

Our aim is first to find the ML permutation $\boldsymbol{\varphi}^* = \boldsymbol{\varphi}^*_{\mathrm{ML}}$, and then to compute the probability that each track is valid. These two steps are repeated for each frame of data until the cumulative probability for each track is sufficiently close to 0 (invalid track) or 1 (valid track). In Section 2.5 an algorithm is given to obtain the ML permutation $\boldsymbol{\varphi}^*_{\mathrm{ML}}$. In Section 2.6 the sequential probability of track validity is computed.

2.5. Score Matrix and Joint Density Maximization

In view of the geometric mean being less than or equal to the arithmetic mean, maximizing the sum of the $p_{j,\boldsymbol{\varphi}(j)}$ does not always maximize the product of the $p_{j,\boldsymbol{\varphi}(j)}$, so does not always maximize the joint density function in Eq. (18). However, the logarithm is monotonic, so we can maximize the logarithm of the joint density function. Hence we have from Eq. (18),

$$\log p(\boldsymbol{Y}\,|\,\boldsymbol{\varphi}) = \sum_{j=1}^{N} [\log p_{j,\boldsymbol{\varphi}(j)}(y_{j,\boldsymbol{\varphi}(j)}) + \log p_{j+N,\boldsymbol{\varphi}(j+N)}(y_{j+N,\boldsymbol{\varphi}(j+N)})] \tag{22}$$

We seek the permutation $\boldsymbol{\varphi}$ that maximizes $\log p(\boldsymbol{Y}\,|\,\boldsymbol{\varphi})$, or equivalently minimizes $-\log p(\boldsymbol{Y}\,|\,\boldsymbol{\varphi})$, thus maximizing the joint density function in Eq. (18). Define the $2N \times (M+N)$ score matrix

$$S = \left[\begin{array}{c|c} S_{1,1} & S_{0,1} \\ \hline S_{1,0} & S_{0,0} \end{array}\right] = \left[\begin{array}{ccc|ccc} s_{1,1} & \cdots & s_{1,M} & s_{1,M+1} & \cdots & s_{1,M+N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{N,1} & \cdots & s_{N,M} & s_{N,M+1} & \cdots & s_{N,M+N} \\ \hline s_{N+1,1} & \cdots & s_{N+1,M} & s_{N+1,M+1} & \cdots & s_{N+1,M+N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{N+N,1} & \cdots & s_{N+N,M} & s_{N+N,M+1} & \cdots & s_{N+N,M+N} \end{array}\right] \tag{23a}$$

where

$$s_{n,m} = -\log p_{n,m} \tag{23b}$$

This score matrix has the structure depicted in the diagram in (24) below. The four events $\{\mathcal{D}\,|\,\mathcal{H}\}$ and the negative logarithm of their likelihoods in Eqs. (14) are depicted in the diagram.[5] The upper left partition is an $N \times M$ matrix of the likelihoods of the test statistics in the event of detection of objects whose tracks are valid. The upper right partition is an $N \times N$ matrix of likelihoods of the test statistics in the event of non-detection of objects whose tracks are valid. The lower left partition is an $N \times M$ matrix of likelihoods of the test statistics in the event of detection when the tracks are invalid. The lower right partition is an $N \times N$ matrix of likelihoods of the test statistics in the event of non-detection when the tracks are invalid.[6]

$$
\begin{array}{cc}
 & \begin{array}{cc} \quad\quad\mathcal{D}_1 \quad\quad\quad\quad & \quad\quad\mathcal{D}_0 \end{array} \\
\begin{array}{c} \mathcal{H}_1 \\ \\ \\ \mathcal{H}_0 \end{array} &
\left(\begin{array}{c|c}
N \times M & N \times N \\
\mathcal{D}_1 \,|\, \mathcal{H}_1 & \mathcal{D}_0 \,|\, \mathcal{H}_1 \\
-\log[\mathcal{N}(\boldsymbol{y}_{j,i})\, P_{\mathrm{D}}] & -\log[\lambda_t (1 - P_{\mathrm{D}})] \\
\hline
N \times M & N \times N \\
\mathcal{D}_1 \,|\, \mathcal{H}_0 & \mathcal{D}_0 \,|\, \mathcal{H}_0 \\
-\log[\lambda_s P_{\mathrm{F}}] & -\log[\lambda_u (1 - P_{\mathrm{F}})]
\end{array}\right)
\end{array}
\tag{24}
$$

The Linear Sum Assignment Problem (LSAP) is to find a permutation vector $\boldsymbol{\varphi}$ corresponding to a permutation matrix $\mathrm{P}_{\boldsymbol{\varphi}}$ that minimizes $\mathrm{tr}(S\,\mathrm{P}_{\boldsymbol{\varphi}})$, where $\mathrm{tr}(\cdot)$ is the trace operator and $\mathrm{P}_{\boldsymbol{\varphi}}$ is the permutation matrix corresponding to the permutation vector $\boldsymbol{\varphi}$. For any permutation $\mathrm{P}_{\boldsymbol{\varphi}}$ of the columns of S,

$$\mathrm{tr}(S\,\mathrm{P}_{\boldsymbol{\varphi}}) = -\log p(\boldsymbol{Y} \,|\, \boldsymbol{\varphi}) \tag{25}$$

The solution $\boldsymbol{\varphi} = \boldsymbol{\varphi}^*$ to the LSAP minimizes Eq. (25) and so is the ML solution[7] to Eq. (21).

There are $(M + N)!$ possible permutations of the columns of the score matrix, so a direct solution to the LSAP (enumerating all permutations) is inefficient and in general is not practical. The LSAP can be solved efficiently by a binary optimization algorithm [31], [9, Ch. 2] such as the Kuhn-Munkres (Hungarian) algorithm [32, 33], Jonker-Volgenant-Castañon (JVC) algorithm [34–36], or the Auction algorithm [37].

When $M \neq N$, an LSAP solver may augment the score matrix with zeros to make it square. This is inconsequential for $M > N$, but affects the solution when $M < N$. We can either expand the right partition to have $\max(0, N - M)$ additional columns or we can eliminate the last $\max(0, N - M)$ rows. These rows have to be restored (but permuted by $\boldsymbol{\varphi}^*$) to obtain the likelihood ratio in Eq. (28).

Remark 1. The score matrix could be defined in terms of the likelihood ratios in Eq. (16) rather than the likelihoods in Eq. (17). This is equivalent to subtracting $-\log[\lambda_s P_{\mathrm{F}}]$ from the columns in the left partition in Eq. (2.5) and subtracting $-\log[\lambda_u(1 - P_{\mathrm{F}})]$ from the columns in the right partition in Eq. (2.5). The log-likelihoods in the lower partition are then all zeros. Although the zero partition could be removed, an LSAP solver will augment the matrix with zeros to make it a square matrix.

Remark 2. Likelihood functions for data of mixed dimensions and units cannot be compared, but non-dimensional likelihood ratios can be compared [7], [9, pp. 128–129]. The likelihood ratio formulation of the score matrix described in Remark 1 would be specified on a per-column basis for mixed measurement types.

Remark 3. The relationship between the maximization of the joint likelihood function (18) and the minimization of the score matrix (23) seems obfuscated in most of the literature. In this work we have shown explicitly the connection between the joint likelihood function and its maximization via the score matrix. Although the joint likelihood function and the score matrix are defined differently from other works, is hoped that the exposition here is a bit easier to follow and provides clarity.

[5] The three rows in each partition are not rows of a matrix. They only serve to describe the hypothesized events and their corresponding likelihood functions.

[6] One could interpret the presence of the lilekihood functions for the $\{\mathcal{D}_0 \,|\, \mathcal{H}_0\}$ hypotheses as a way to maximize the probability of non-detection for invalid tracks.

[7] A modification of the score matrix is needed to obtain the MAP solution in Eq. (20) if $P(\boldsymbol{\varphi})$ is known.

Remark 4. An "augmented" score matrix defined in [2], [4, pp. 249, 276–279], [5, §9.5.3] models multiple pseudo-tracks (new tracks) and only one pseudo-measurement in [4, 5]. After some subtractions from the score matrix, a zero column is deleted. Our formulation of the score matrix in Eq. (23) based on Eqs. (17), (18), and (22) is more general.

2.6. Likelihood Ratio for False Track Detection

In this section we compute the probability that each track (catalog object) is valid. We first obtain, for each track, the likelihood ratio (LR) of the probability density functions under the hypotheses $\mathcal{H} = \mathcal{H}_1$ that the track is valid and $\mathcal{H} = \mathcal{H}_0$ that it is invalid. The LR is computed for the detection events selected by the LSAP. The probability that the track is valid is computed from the LR. The LR is updated sequentially from each frame of data and tested against stopping criteria.

From Bayes' Rule we have the posterior probability of validity of a track is

$$P(\mathcal{H}\,|\,\boldsymbol{y}, \mathcal{D}) = \frac{p(\boldsymbol{y}, \mathcal{D}\,|\,\mathcal{H})\,P(\mathcal{H})}{p(\boldsymbol{y}, \mathcal{D})} \tag{26}$$

where $p(\boldsymbol{y}, \mathcal{D})$ is the unconditional density function of the data. From Eq. (26) and Eq. (11), the likelihood ratio of posterior probabilities of track validity ($\mathcal{H} = \mathcal{H}_1$) and invalidity ($\mathcal{H} = \mathcal{H}_0$) is

$$\begin{aligned}\Lambda = \frac{P(\mathcal{H}_1\,|\,\boldsymbol{y}, \mathcal{D})}{P(\mathcal{H}_0\,|\,\boldsymbol{y}, \mathcal{D})} &= \frac{p(\boldsymbol{y}, \mathcal{D}\,|\,\mathcal{H}_1)\,P(\mathcal{H}_1)}{p(\boldsymbol{y}, \mathcal{D}\,|\,\mathcal{H}_0)\,P(\mathcal{H}_0)} \\ &= \frac{p(\boldsymbol{y}\,|\,\mathcal{D}, \mathcal{H}_1)\,P(\mathcal{D}\,|\,\mathcal{H}_1)\;P(\mathcal{H}_1)}{p(\boldsymbol{y}\,|\,\mathcal{D}, \mathcal{H}_0)\,P(\mathcal{D}\,|\,\mathcal{H}_0)\;P(\mathcal{H}_0)}\end{aligned} \tag{27}$$

Note that $p(\boldsymbol{y}, \mathcal{D})$ cancels out in the likelihood ratio. At frame k, the prior probability that the track is valid is $P_{k-1}(\mathcal{H}_1)$ and the prior probability that the track is invalid is $P_{k-1}(\mathcal{H}_0)$. Setting $\boldsymbol{y} = \boldsymbol{y}^k_{j,\boldsymbol{\varphi}^*(j)}$ in Eq. (27), the likelihood ratio Λ_k for track j at time k is

$$\begin{aligned}\Lambda_k &= \frac{p(\boldsymbol{y}^k_{j,\boldsymbol{\varphi}^*(j)}\,|\,\mathcal{D}, \mathcal{H}_1)\,P_{k-1}(\mathcal{D}\,|\,\mathcal{H}_1)\,P_{k-1}(\mathcal{H}_1)}{p(\boldsymbol{y}^k_{j+N,\boldsymbol{\varphi}^*(j)}\,|\,\mathcal{D}, \mathcal{H}_0)\,P_{k-1}(\mathcal{D}\,|\,\mathcal{H}_0)\,P_{k-1}(\mathcal{H}_0)} \\ &= \frac{p^k_{j,\boldsymbol{\varphi}^*(j)}}{p^k_{j+N,\boldsymbol{\varphi}^*(j)}}\,\frac{P_{k-1}(\mathcal{H}_1)}{P_{k-1}(\mathcal{H}_0)}\,, \qquad 1 \le j \le N\end{aligned} \tag{28}$$

The final result in Eq. (28) follows from the density functions in Eqs. (14) and the structure of the score matrix in (24). To conserve notation, Λ_k is not tagged with the track number j and the detection event $\mathcal{D}$ is not tagged with j and k. From Eq. (28), the likelihood ratio is updated with each frame of data as a product of the current and prior likelihood ratios [30],

$$\Lambda_k = c_k \Lambda_{k-1} \tag{29}$$

where, for the jth track,

$$c_k = \frac{p^k_{j,\boldsymbol{\varphi}^*(j)}}{p^k_{j+N,\boldsymbol{\varphi}^*(j)}} = \begin{cases} \dfrac{\mathcal{N}(\boldsymbol{y}^k_{j,\boldsymbol{\varphi}^*(j)}; \boldsymbol{0}, \Sigma_{j,\boldsymbol{\varphi}^*(j)})\,P_\text{D}}{\lambda_s P_\text{F}} & \mathcal{D} = \mathcal{D}_1 \quad (1 \le \boldsymbol{\varphi}^*(j) \le N) \\[2ex] \dfrac{\lambda_t(1 - P_\text{D})}{\lambda_u(1 - P_\text{F})} & \mathcal{D} = \mathcal{D}_0 \quad (M + 1 \le \boldsymbol{\varphi}^*(j) \le M + N) \end{cases} \tag{30}$$

Equivalently, for each track j the c_k is computed from the (j, j) and $(j + N, j)$ elements of the permuted score matrix $S\,\mathrm{P}_{\boldsymbol{\varphi}}$. Practical bounds on the densities λ_s, λ_t and λ_u are derived in Section 2.7.

Remark 5. The likelihood ratio c_k is the term on the right side of Eq. (16).

The assignment algorithm and likelihood ratios are computed for each frame k of measurements. At iteration k, the prior probabilities are the posterior probabilities from the previous iteration,

$$P_{k-1}(\mathcal{H}_1) = P(\mathcal{H}_1 \,|\, \boldsymbol{y}^{k-1}_{j,\boldsymbol{\varphi}^*(j)}, \mathcal{D}) \tag{31a}$$

$$P_{k-1}(\mathcal{H}_0) = P(\mathcal{H}_0 \,|\, \boldsymbol{y}^{k-1}_{j+N,\boldsymbol{\varphi}^*(j)}, \mathcal{D}) \tag{31b}$$

In view of Eq. (26), computation of these probabilities require computation of $p(\boldsymbol{y}, \mathcal{D})$. This density is unknown but could be computed using the Total Probability Theorem. However, its computation can be avoided as we will see momentarily. Since the hypotheses $\mathcal{H}_1$ and $\mathcal{H}_0$ are binary, the likelihood ratio in Eq. (28) is the ratio of complementary probabilities such that $P_k(\mathcal{H}_0) = 1 - P_k(\mathcal{H}_1)$, so we have

$$\Lambda_k = \frac{P_k(\mathcal{H}_1)}{P_k(\mathcal{H}_0)} = \frac{P_k(\mathcal{H}_1)}{1 - P_k(\mathcal{H}_1)} \tag{32}$$

Solving this equation for $P_k(\mathcal{H}_1)$ yields the probability that the track is valid,

$$P_k(\mathcal{H}_1) = \frac{\Lambda_k}{1 + \Lambda_k} \tag{33}$$

Computation of $p(\boldsymbol{y}, \mathcal{D})$ in Eq. (26) is thus circumvented by computing $P_k(\mathcal{H}_1)$ from the likelihood ratio.

The initial likelihood ratio is

$$\Lambda_0 = \frac{P_0(\mathcal{H}_1)}{P_0(\mathcal{H}_0)} = \frac{P_0(\mathcal{H}_1)}{1 - P_0(\mathcal{H}_1)} \tag{34}$$

If we assume (initially) equal probability that the track is valid or invalid, or if we lack reliable information for the prior probabilities, we can choose agnostic prior probabilities $P_0(\mathcal{H}_1) = P_0(\mathcal{H}_0) = 0.5$ for which $\Lambda_0 = 1$. Prior probabilities based on reliable information can, on average, result in quicker decisions.

The hypotheses are accepted or rejected depending on the value of the likelihood ratio compared to decision thresholds. The decision thresholds are[8]

$$A = \frac{1 - \alpha}{\beta} \qquad B = \frac{\alpha}{1 - \beta} \tag{35}$$

where the parameter α is the probability of false alarm (Type I Error) and β is the probability of missed detection (Type II Error), so $1 < A < \infty$ and $0 < B < 1$. These are referred to, respectively, as the false track confirmation probability and the true track deletion probability. The stopping criteria for the SPRT are

$$\left.\begin{array}{rl} \Lambda_k \geq A & \text{accept } \mathcal{H}_1 \\ \Lambda_k \leq B & \text{accept } \mathcal{H}_0 \\ B < \Lambda_k < A & \text{take more data or decide with probability } P_k(\mathcal{H}_1) \end{array}\right\} \tag{36}$$

The log-likelihood ratio is used in the false-object detection as in [6]. Since the negative log-likelihoods are available from the score matrix, the log-likehood ratio $\mathcal{L}_k = \log \Lambda_k$ is updated according to

$$\begin{aligned} \mathcal{L}_k &= \mathcal{L}_{k-1} + \log p(\boldsymbol{y}^k_{j,\boldsymbol{\varphi}^*(j)}, \mathcal{D} \,|\, \mathcal{H}_1) - \log p(\boldsymbol{y}^k_{j+N,\boldsymbol{\varphi}^*(j)}, \mathcal{D} \,|\, \mathcal{H}_0) \\ &= \mathcal{L}_{k-1} + \log p^k_{j,\boldsymbol{\varphi}^*(j)} - \log p^k_{j+N,\boldsymbol{\varphi}^*(j)} \\ &= \mathcal{L}_{k-1} + \log c_k \end{aligned} \tag{37}$$

[8]Customarily the null hypothesis $\mathcal{H}_0$ is in the numerator. The consequence of putting $\mathcal{H}_1$ in the numerator is that one has to be careful in interpreting and applying the false alarm and false detection probabilities α and β in Eq. (35).

Since entries in the score matrix are the negative logarithms of the densities, we have simply

$$\log c_k = -s_{j,\boldsymbol{\varphi}^*(j)} + s_{j+N,\boldsymbol{\varphi}^*(j)} \tag{38}$$

Thus $\mathcal{L}_{k-1}$ is updated simply by indexing two elements of the score matrix S and an addition and a subtraction.

Now, from Eq. (33) we have

$$P_k(\mathcal{H}_1) = \frac{\exp(\mathcal{L}_k)}{1 + \exp(\mathcal{L}_k)} \tag{39}$$

From Eq. (34) the initial log-likelihood ratio is

$$\mathcal{L}_0 = \log \Lambda_0 = \log \frac{P_0(\mathcal{H}_1)}{P_0(\mathcal{H}_0)} = \log \frac{P_0(\mathcal{H}_1)}{1 - P_0(\mathcal{H}_1)} \tag{40}$$

For equally likely prior probabilities, we have $P(\mathcal{H}_0) = P(\mathcal{H}_1) = 0.5$ for which $\mathcal{L}_0 = 0$.

The stopping criteria for the log-likelihood ratio test are

$$\left.\begin{array}{rl} \mathcal{L}_k \geq a & \text{accept } \mathcal{H}_1 \\ \mathcal{L}_k \leq b & \text{accept } \mathcal{H}_0 \\ b < \mathcal{L}_k < a & \text{take more data or decide with probability } P_k(\mathcal{H}_1) \end{array}\right\} \tag{41}$$

where $a = \log A$ and $b = \log B$. Observe that $a > 0$ and $b < 0$. For $\alpha = \beta$ the logarithmic thresholds are symmetric with $b = -a$.

We could base the stopping criteria on a probability threshold test

$$\left.\begin{array}{rl} P_k(\mathcal{H}_1) \geq 1 - P_t & \text{accept } \mathcal{H}_1 \\ P_k(\mathcal{H}_1) \leq P_t & \text{accept } \mathcal{H}_0 \\ P_t < P_k(\mathcal{H}_1) < 1 - P_t & \text{take more data or decide with probability } P_k(\mathcal{H}_1) \end{array}\right\} \tag{42}$$

where P_t is a small probability threshold. The probability threshold test is equivalent to the tests in Eq. (36) and Eq. (41) with $\alpha = \beta = P_t$.

Remark 6. The development in this section is similar to the track deletion algorithm in [5, §6.2] and [6]. The main difference is that the definition of c_k in Eq. (28) and Eq. (30) with $\mathcal{D} = \mathcal{D}_0$ is more general than in [5, Eq. (6.10b)].

2.7. Bounds on Spurious Measurement and False Track Densities

The spurious measurement and target densities λ_s and λ_t are chosen in [5, 7, 8, 11] based on knowledge of the actual density of spurious measurements and target densities, specified in terms of numbers of spurious measurements and targets in the surveillance volume. Such information may be difficult to obtain and varies over time and space. In the following we obtain bounds on λ_s, λ_t, and λ_u to ensure proper operation of the track validation algorithm. For convenience and legibility, let $\boldsymbol{y} = \boldsymbol{y}_{j,\boldsymbol{\varphi}(j)}$ and $\Sigma = \Sigma_{j,\boldsymbol{\varphi}(j)}$ in what follows.

A track is detected under the hypothesis $\mathcal{H} = \mathcal{H}_1$ when the likelihood ratio

$$\mathcal{N}(\boldsymbol{y}; \mathbf{0}, \Sigma)\, P_\mathrm{D} \,/\, \lambda_t(1 - P_\mathrm{D}) > 1 \tag{43}$$

for a desired χ^2 probability threshold. Thus the likelihood threshold is set by choosing λ_t so that

$$\lambda_t \leq [P_\mathrm{D}/(1 - P_\mathrm{D})] \,/\, [(2\pi)^d |\Sigma| e^\tau]^{1/2} \tag{44}$$

for τ corresponding to some small probability that $\boldsymbol{y}^T\Sigma^{-1}\boldsymbol{y} > \tau$, which can be can be computed or found in a χ^2 table. An upper bound on $|\Sigma|$ could be determined to obtain a single value for λ_t.

Consider the likelihood ratio labeled $\mathcal{D}_1$ in Eq. (30). In the case of detection ($\mathcal{D} = \mathcal{D}_1$) for which a measurement-track association occurs ($\mathcal{H} = \mathcal{H}_1$), we must have

$$\mathcal{N}(\boldsymbol{y};\boldsymbol{0},\Sigma)\, P_{\mathrm{D}} \,/\, \lambda_s P_{\mathrm{F}} > 1 \tag{45}$$

so that the sequential likelihood ratio increases when $\mathcal{H} = \mathcal{H}_1$. Thus we require

$$\lambda_s \leq (P_{\mathrm{D}}/P_{\mathrm{F}}) \,/\, \left[(2\pi)^d|\Sigma|e^{\tau}\right]^{1/2} \tag{46}$$

for τ corresponding to some small probability that $\boldsymbol{y}^T\Sigma^{-1}\boldsymbol{y} > \tau$. An upper bound on $|\Sigma|$ could be determined to obtain a single value for λ_s.

Now consider the likelihood ratio labeled $\mathcal{D}_0$ in Eq. (30). In the case of non-detection ($\mathcal{D} = \mathcal{D}_0$) for which a measurement-track association does not occur ($\mathcal{H} = \mathcal{H}_0$), we must have

$$\frac{\lambda_t(1-P_{\mathrm{D}})}{\lambda_u(1-P_{\mathrm{F}})} < 1 \tag{47}$$

so that the sequential likelihood ratio decreases. Thus we require that

$$\lambda_u > \lambda_t\left(\frac{1-P_{\mathrm{D}}}{1-P_{\mathrm{F}}}\right) \tag{48}$$

We normally have $P_{\mathrm{F}} \ll P_{\mathrm{D}}$ and typically $P_{\mathrm{F}} \leq 1 - P_{\mathrm{D}}$, so we have $\lambda_t \leq \lambda_s$ for the same τ in Eq. (44) and Eq. (46). Setting $\lambda_u = \lambda_t$ gives the likelihood ratio update in [5, p. 330, Eq. (6.10b)] and the log-likelihood ratio update in [6, Eq. 5].

The results of the next section provide an alternate, or supplemental, means for choosing λ_s, λ_t, and λ_u.

2.8. Track Valid/Invalid Decision Time

Assume that the detection decision is $\mathcal{D}_0$ on every update of the likelihood ratio Λ_k, and assume that the inequality in Eq. (47) holds. The validation test stops after n_0 updates and declares $\mathcal{H}_0$ (invalid track) when

$$\left(\frac{\lambda_t(1-P_{\mathrm{D}})}{\lambda_u(1-P_{\mathrm{F}})}\right)^{n_0}\Lambda_0 \leq B \tag{49}$$

The minimum number of updates with the test terminating with the decision $\mathcal{H} = \mathcal{H}_0$ (invalid track) is

$$n_0 = \left\lceil \log(B/\Lambda_0) \,/\, \log\left(\frac{\lambda_t(1-P_{\mathrm{D}})}{\lambda_u(1-P_{\mathrm{F}})}\right)\right\rceil \tag{50}$$

where $\lceil q \rceil$ is the smallest integer greater than or equal to q.

Now assume that the detection decision is $\mathcal{D}_1$ on every update of the likelihood ratio Λ_k, and assume that the inequality in Eq. (46) holds. For simplicity assume that $\Sigma = \sigma^2 I_d$. (We have also omitted the subscripts j and $\boldsymbol{\varphi}(j)$.) The expected value of the log-likelihood under the hypothesis $\{\mathcal{D}_1\,|\,\mathcal{H}_1\}$ is

$$\begin{aligned}
\mathcal{E}\left\{\log\mathcal{N}(\boldsymbol{y};\boldsymbol{0},\sigma^2 I)\right\} &= -\log\left[(2\pi\sigma^2)^{d/2}\right] - \tfrac{1}{2}\,\mathcal{E}\{\chi^2\} \\
&= -(d/2)\log(2\pi\sigma^2) - d/2 \\
&= -(d/2)\log(2\pi\sigma^2 e)
\end{aligned} \tag{51}$$

where $\chi^2 = \boldsymbol{y}^T \Sigma^{-1} \boldsymbol{y}$ is a chi-square random variable with d degrees of freedom. The validation test stops after n_1 updates (on average) and declares $\mathcal{H}_1$ (valid track) when

$$\left(\frac{P_\mathrm{D}}{(2\pi\sigma^2 e)^{d/2} \lambda_s P_\mathrm{F}} \right)^{n_1} \Lambda_0 \geq A \tag{52}$$

The average minimum number of updates with the test terminating with the decision $\mathcal{H} = \mathcal{H}_1$ (valid track) is

$$n_1 = \left\lceil \log(A/\Lambda_0) \,/\, \log\left(\frac{P_\mathrm{D}}{(2\pi\sigma^2 e)^{d/2} \lambda_s P_\mathrm{F}} \right) \right\rceil \tag{53}$$

The number of updates (with one update per track per frame of data) required to decide $\mathcal{H} = \mathcal{H}_0$ or $\mathcal{H} = \mathcal{H}_1$ can be greater than n_0 and n_1 due to random error in $\boldsymbol{y}$ as well as missed detections. In dense scenarios, n_0 and n_1 can significantly underestimate the number of updates required to decide the validity of the tracks. In some scenarios, random error and missed detections can result in an incorrect decision.

The decision time is increased, and the sensitivity of the likelihood ratio to association errors is reduced, when c_k in Eq. (30) is closer to 1. Thus the track validation test can be made to be more reliable in dense object scenarios and crossing tracks by setting λ_s, λ_t, and λ_u appropriately. Although guidelines for choosing λ_s, λ_t, and λ_u were given in Section 2.7, Eq. (49) and Eq. (52) provide an alternate, or supplemental, means for choosing these parameters. Set $\Lambda_0 = 1$ and choose n_0 and n_1 (with $n_0 = n_1$ trading flexibility for simplicity). Compute λ_t using Eq. (44) and then compute λ_u from Eq. (49). Finally, compute λ_s from Eq. (52). Then check that the inequalities in Eq. (46) and Eq. (48) are satisfied.

2.9. Ambiguity Detection and Resolution

Murty's m-Best assignment algorithm [38–43], [9, Ch. 2] can be used to find the m largest joint densities $p(\boldsymbol{Y}|\boldsymbol{\varphi}^\ell)$ and their corresponding permutations $\boldsymbol{\varphi}^\ell$, $1 \leq \ell \leq m$, such that

$$p(\boldsymbol{Y} \,|\, \boldsymbol{\varphi}^1) \geq p(\boldsymbol{Y} \,|\, \boldsymbol{\varphi}^2) \geq \cdots \geq p(\boldsymbol{Y} \,|\, \boldsymbol{\varphi}^m) \tag{54}$$

Then from (19) (with equal prior probabilities $P(\boldsymbol{\varphi}^\ell)$ for each permutation $\boldsymbol{\varphi}^\ell$), the posterior assignment probabilities are ordered

$$P(\boldsymbol{\varphi}^1 \,|\, \boldsymbol{Y}) \geq P(\boldsymbol{\varphi}^2 \,|\, \boldsymbol{Y}) \geq \cdots \geq P(\boldsymbol{\varphi}^m \,|\, \boldsymbol{Y}) \tag{55}$$

The probability of error in the MAP or ML assignment $\boldsymbol{\varphi}^* = \boldsymbol{\varphi}^1$ is large when $\boldsymbol{\varphi}^2 \simeq \boldsymbol{\varphi}^1$, that is, when the second largest posterior probability is close to the largest posterior probability.[9] (For ML, we can compare density functions.) This occurs when tracks cross or when when two or more tracks are not well separated compared to the prediction errors and measurement errors. That is, two or more error ellipsoids of residuals overlap with significant probability, which increases the uncertainty in the estimated optimal assignment.

The likelihood ratio test can incorrectly validate or invalidate a track (a catalog object) in the presence of ambiguity. One remedy is to skip the update of the likelihood ratio in Eq. (29) or of the log-likelihood ratio in Eq. (37) when the assignment probabilities are close (within a factor of 10, for example) until the data changes and the ambibuity is resolved.

Rather than skipping an update of the likelihood ratio (or its logarithm), we can modify the update according to the closeness of the largest and second largest assignment probabilities. For simplicity, assume

[9] It is well known in multiple hypothesis testing that the probability of error in the optimal decision is large when the density functions are "close" in some sense [44–46]. Unfortunately the probability of error can be computed only approximately for multivariate densities.

the $P(\boldsymbol{\varphi}^\ell)$ are all equal. Since $P(\boldsymbol{\varphi}^1) \geq P(\boldsymbol{\varphi}^2)$, we have that

$$0 \leq \frac{P(\boldsymbol{\varphi}^2 \mid \boldsymbol{Y})}{P(\boldsymbol{\varphi}^1 \mid \boldsymbol{Y})} \leq 1 \tag{56}$$

Since $p(\boldsymbol{Y})$ cancels in the ratio of the probabilities (cf. Eq. (19)), we have

$$0 \leq \frac{p(\boldsymbol{\varphi}^2 \mid \boldsymbol{Y})}{p(\boldsymbol{\varphi}^1 \mid \boldsymbol{Y})} \leq 1 \tag{57}$$

Since the m probabilities $P(\boldsymbol{\varphi}^\ell \mid \boldsymbol{Y})$ sum to 1 and $P(\boldsymbol{\varphi}^1 \mid \boldsymbol{Y})$ is the largest, we have that $P(\boldsymbol{\varphi}^1 \mid \boldsymbol{Y}) > 0$. We now define an association ambiguity factor

$$\theta = \frac{p(\boldsymbol{\varphi}^2 \mid \boldsymbol{Y})}{p(\boldsymbol{\varphi}^1 \mid \boldsymbol{Y})} \tag{58}$$

This is used to interpolate between 1 and c_k to obtain a modified likelihood update factor

$$\begin{aligned} c'_k &= (1-\theta)c_k + \theta \\ &= c_k + \theta(1 - c_k) \end{aligned} \tag{59}$$

The update of the likelihood ratio (or the log-likelihood ratio) for each track is given by Eq. (29) (or Eq. (37)) with c_k replaced by c'_k. When $P(\boldsymbol{\varphi}^2) \ll P(\boldsymbol{\varphi}^1)$, the ambiguity is small, $\theta \simeq 0$, and $c'_k \simeq c_k$. When $P(\boldsymbol{\varphi}^2) \simeq P(\boldsymbol{\varphi}^1)$, the ambiguity is large, $\theta \simeq 1$, and $c'_k \simeq 1$. The interpolation is illustrated in Figure 1 for c_k ranging from 0 to 2 and θ ranging from 0 to 1. (In the graph we show $c'_k = f(c, \theta)$ for clarity.) Although somewhat ad hoc, this modification of the likelihood update will give a more reliable track validation or invalidation in the presence of ambiguous assignments.

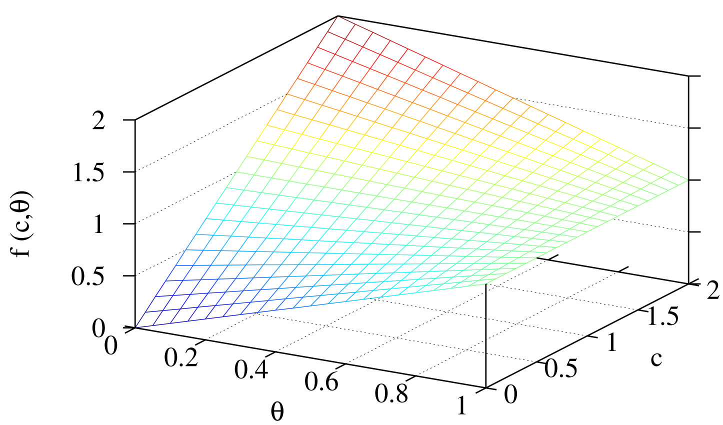


Fig. 1: Interpolation of c_k for the association ambiguity factor θ ranging from 0 to 1.

3. SIMULATION RESULTS

The parameters for the false object identification algorithm are listed in Table 1. For the purpose of illustration we use a simple state and observation model. We have for each tracked object the 2×1 position vector $\boldsymbol{x}_n$, the observation matrix $H_n = I_{2\times2}$, the position state covariance $S_j = \sigma_\mathrm{p}^2 I_{2\times2}$ with $\sigma_\mathrm{p} = 0.96$, and the measurement covariance $R_i = \sigma_\mathrm{m}^2 I_{2\times2}$ with $\sigma_\mathrm{m} = 0.45$. The total error is therefore $\sigma_e = (\sigma_\mathrm{p}^2 + \sigma_\mathrm{m}^2)^{1/2} = 1.06$. The observation is linear in the state with $\boldsymbol{h}_i(\boldsymbol{x}_j) = \boldsymbol{x}_j$ and $H_i = I_{2\times2}$. Other parameters are listed in Table 1. A Matlab implementation of the JVC algorithm [36] was used to compute the data association. The simulator and the false object identification algorithm were implemented in Octave.

Table 1: Parameters

Quantity	Value	Definition
σ_p	0.96	standard deviation of state estimate prediction error
σ_m	0.45	standard deviation of measurement error
α	0.01	false (invalid) track confirmation probability
β	0.01	true (valid) track deletion probability
P_t	0.01	true (valid) track deletion probability
P_D	0.8	probability of detection
P_F	0.05	probability of false alarm
$P_0(\mathcal{H}_1)$	0.5	initial prior probability that a track is valid
λ_s	0.131	spurious measurement density
λ_t	0.066	valid object density
λ_u	0.131	invalid object density

The simulated tracking scenario and the computed association scores, likelihood ratios, and probabilities are shown in Figure 2, Figure 3, and Figure 4. The scenarios shown in Figure 2 and Figure 4 comprise three parallel tracks and one crossing track. The scenario in Figure 3 is similar but comprises five parallel tracks and one crossing track. In each scenario the parallel tracks are separated 3 units vertically, so the perpendicular separation distances are 2.12 units ($= \sigma_e$ at the midpoint between the tracks). The notional space objects move in each direction with constant velocity of 1 unit per measurement interval. Measurements are (x, y) coordinate positions (notional angle measurements). The middle track (#4 in Figure 2 and #6 in Figure 3) and crossing track (#3 in Figure 2 and #5 in Figure 3) are not measured.

Numerical results are shown in Table 2 for a single scenario and for 100 Monte Carlo (MC) iterations of the same scenario but different random errors in positions and measurements. Results for three scenarios are shown. The data shown are the number of iterations until convergence (of probability of valid track) and percentage of correct associations are shown for a single-run and MC runs, and the average number of correct valid/invalid-track decisions are shown. The valid-track probability is tested using Eq. (42).

Although the middle track is not measured, about 6% of the time in Scenario 1 and 3% of the time in Scenario 2 it "steals" enough measurements to appear as a valid track, that is, the track valid/invalid decision is correct 94% (resp. 97%) of the time. Valid/invalid decisions for the other tracks are nearly always correct. The percentage of correct track-measurement associations (and non-associations) are correct with almost 90% probability. Convergence to a decision threshold occurs in 5–8 iterations for most tracks but is a bit longer for the middle track. The results for a single scenario are graphed in Figure 2 and Figure 3. The overall results for Scenario 1 and Scenario 2 are similar.

Table 2: False Object Identification Results

Track	Convergence (iterations)		Correct associations (%)		Correct decisions (%)
	scenario	mc average	scenario	mc average	mc average (%)
Scenario 1					
1	4.0	8.0	83.3	87.4	100.0
2	6.0	8.2	91.7	86.6	99.0
3	4.0	5.5	91.7	91.4	100.0
4	12.0	11.8	83.3	86.2	94.0
Scenario 2					
1	4.0	8.0	88.9	91.1	98.0
2	4.0	8.8	100.0	83.8	100.0
3	9.0	8.5	77.8	86.7	100.0
4	6.0	10.0	100.0	92.1	100.0
5	6.0	7.4	77.8	87.3	100.0
6	4.0	11.3	100.0	87.4	97.0
Scenario 3					
1	15.0	12.4	81.0	83.6	98.0
2	11.0	13.5	81.0	82.5	97.0
3	7.0	5.4	95.2	94.4	100.0
4	21.0	22.4	71.4	75.6	79.0

The scenario in Scenario 3 is similar to that in Scenario 1 except that the track prediction error and measurement error are increased by 25%. The results for Scenario 3 in Table 2 show that the convergence time is essentially doubled. The percentage of correct associations and correct decisions decreases considerably for the middle track. It can be shown that increasing the state estimate and measurement errors drives the decision error toward 50% for the middle track and increase the decision error for the other tracks as well. The track-measurement association is of course also affected as the state estimate and measurement errors increase.

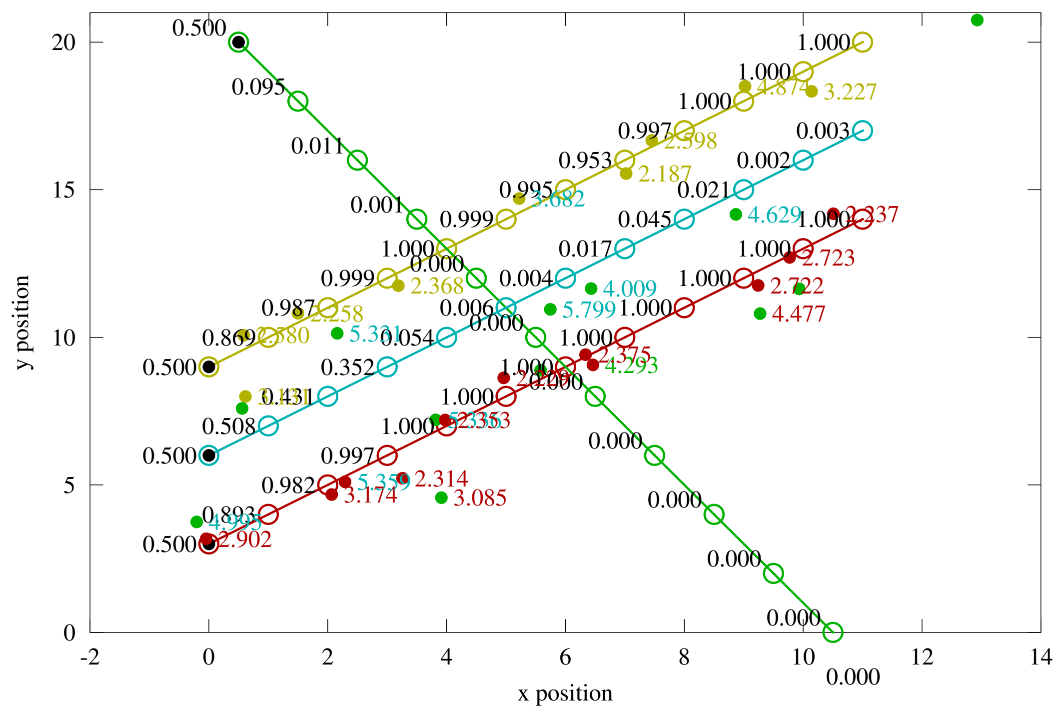
4. CONCLUSION

The track validation algorithm is designed to assist in maintenance of a catalog of tracked space objects for Space Situational Awareness. Invalid tracks (estimated orbits stored in the catalog) produced by a tracking system or by an analyst have to be identified and removed from the catalog. The track validation algorithm computes the probability that a track is valid or invalid based on observation data, comprising one or more of angle pairs, range, Doppler measurements, and attribute or feature data. The track validation algorithm first computes the maximum likelihood permutation of probability density functions of the data to obtain associations and non-associations of tracks and measurements, and then it computes the probabilities that the tracks are valid. These two steps are repeated for each frame of data and each cumulative probability is computed until it is sufficiently close to 0 (invalid object) or 1 (valid object).

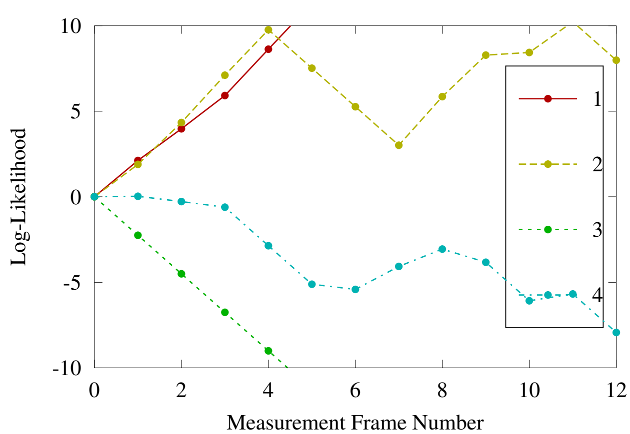
The track-measurement association algorithm and the track validation algorithm developed in this work differs from developments found in the literature. We avoid the subterfuges “pseudo-measurements”, “pseudo-tracks”, or “dummy-tracks” defined in other works; these are euphamisms for the null and alternate hypotheses defined in Section 2.2 and Section 2.3. The joint probability in Eq. (18) that we maximize and the formulation of the score matrix in Eq. (23) differ from other works. We show explicitly in Sections 2.4 and 2.5 how the solution to the linear sum assignment problem (LSAP) provides the MAP or ML assignments (or non-assignments) of tracks and measurements. The structure of the score matrix is based on the four combinations of hypotheses for detection, non-detection, valid track, and invalid track. The entries in the score matrix are the negative logarithm of the probability densities of the data under the four combinations of these hypotheses for each track.

The performance of track-measurement association algorithm and the track validation algorithm is set by the spurious measurement density λ_s, the track density λ_t, and the invalid track density λ_u, as well as the data, statistical parameters of the data, and sensor performance. Formulas to calculate suitable values for λ_s, λ_t, and λ_u were derived to ensure proper operation of the algorithm. These parameters determine the association thresholds and can be chosen to set the convergence time and sensitivity of the probability of track validity to the data, which is useful particularly in the event of close and crossing tracks. The association ambiguity algorithm in Section 2.9 has not yet been tested, but it should improve performance in dense object scenarios and avoid incorrect track confirmations and deletions.

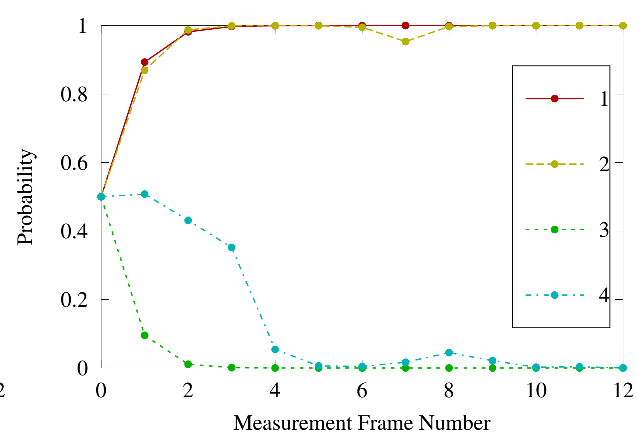
Monte Carlo simulation results are given for three dense tracking scenarios comprising parallel tracks and a crossing track, where a middle track and the crossing track are not measured and so are invalid. The observation data comprise a pair of angle measurements of each of the tracked space objects. The simulation results show that the tracks are validated or invalidated correctly with 94% to 100% probability in 6 to 12 iterations on average. In a more dense tracking scenario, the invalid middle track was deemed invalid in 79% of cases in 6 to 22 iterations on average.



a. Tracks with probabilities of valid track (black) and association scores (color). Incorrect associations are shown with different colors for the marker and score.

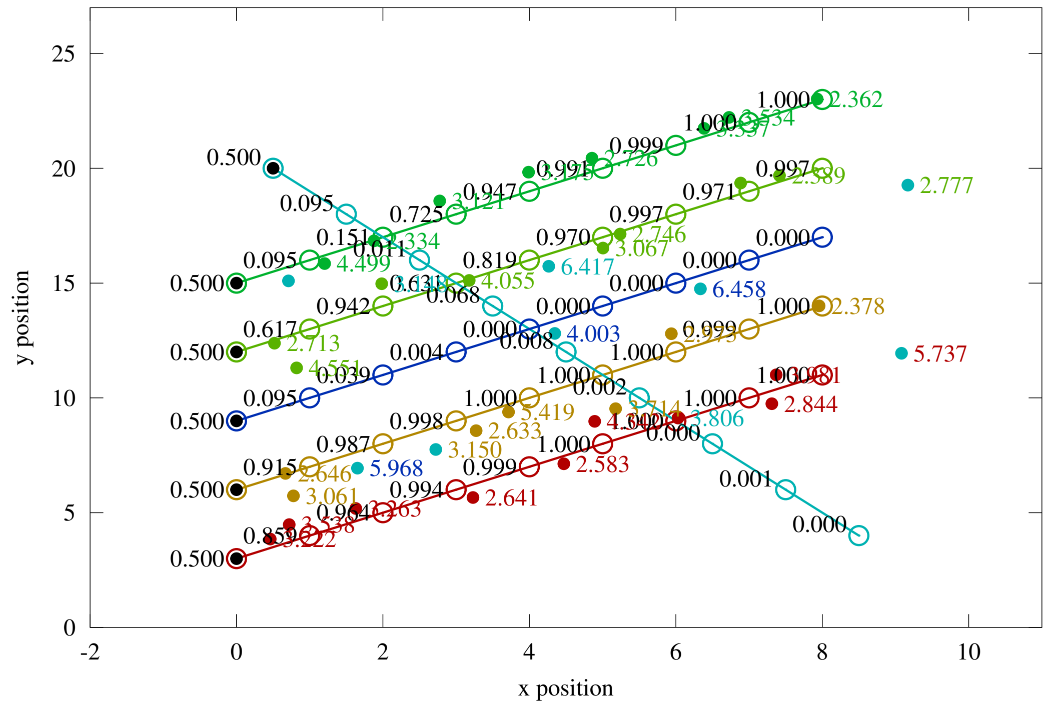


b. Log-likelihood ratio for each track.

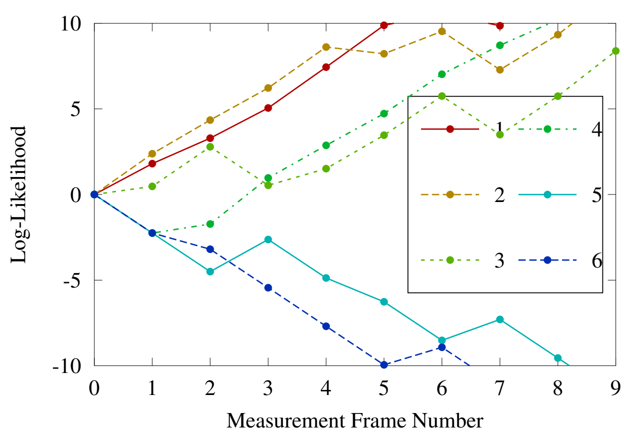


c. Probability of valid track.

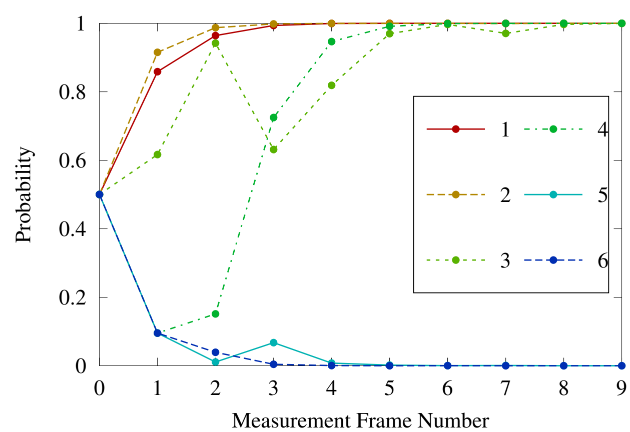
Fig. 2: Scenario 1. Track-measurement association results for a four-track scenario. Middle track and crossing track are not measured.



a. Tracks with probabilities of valid track (black) and association scores (color). Incorrect associations are shown with different colors for the marker and score.

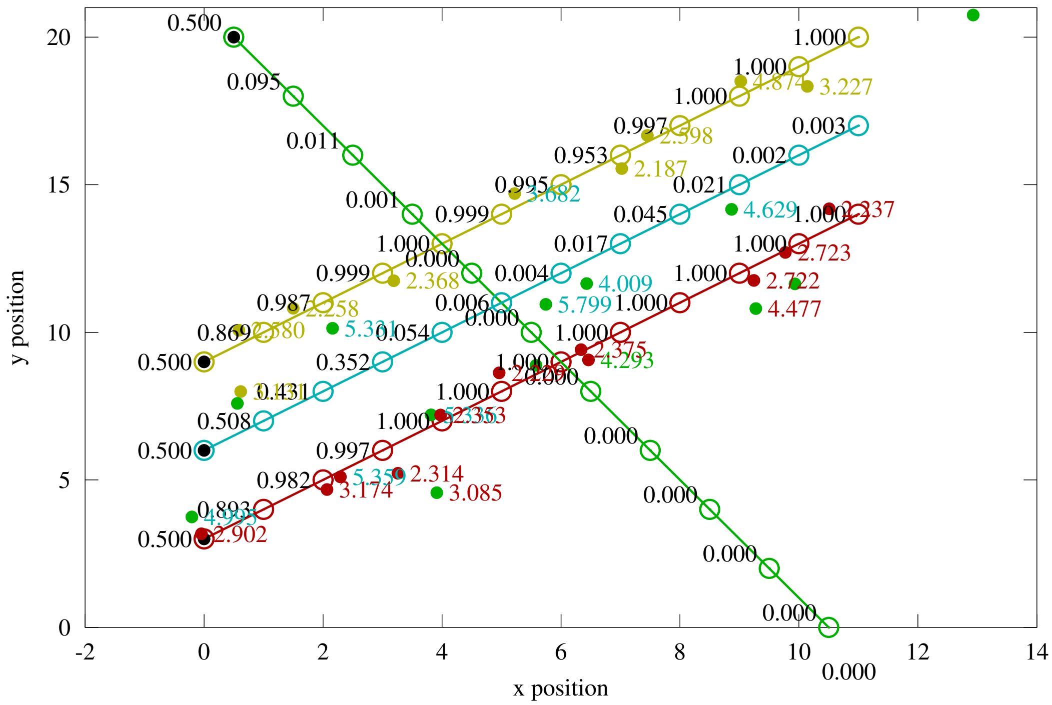


b. Log-likelihood ratio for each track.

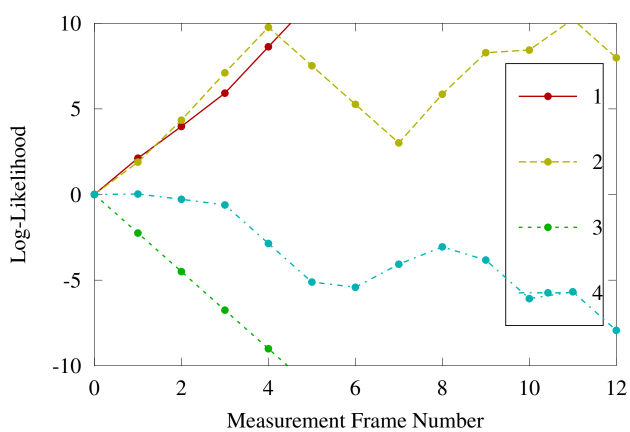


c. Probability of valid track.

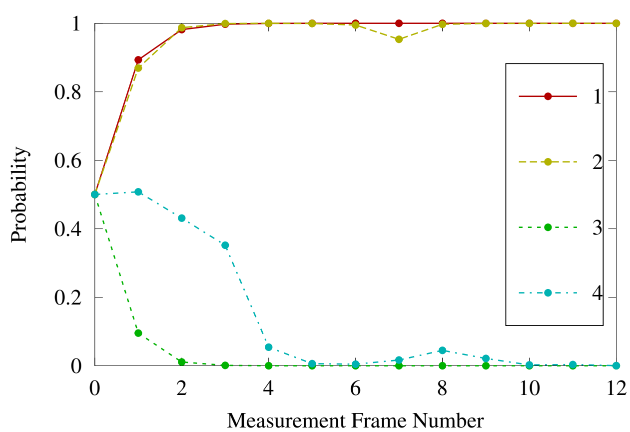
Fig. 3: Scenario 2. Track-measurement association results for a six-track scenario. Middle track and crossing track are not measured.



a. Tracks with probabilities of valid track (black) and association scores (color). Incorrect associations are shown with different colors for the marker and score.



b. Log-likelihood ratio for each track.



c. Probability of valid track.

Fig. 4: Scenario 3. Track-measurement association results for a four-track scenario. Middle track and crossing track are not measured. State estimate and measurement errors are increased by 25%.

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