Applications of Random Sampling Consensus to Space Object Motion Analysis

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1. ABSTRACT

This paper explores the Random Sampling Consensus [1], (RANSAC) algorithm as a methodology for data association and illustrates its direct integration with angles-only Initial Orbit Determination (IOD). The resulting Consensus Sets of measurements are analyzed using a connectivity matrix which indicates the membership between measurements and best-fitting IOD states which meet certain quality criteria. Using this metric, it is demonstrated how potential outliers and miss-tags in input data can be identified and discarded from subsequent precision orbit determination processes. Finally, challenging input datasets which include violations of the assumptions of the motion estimation process will be analyzed, demonstrating the utility of this method as an indicator of complex behavior contained within collected data.

2. INTRODUCTION

Generating meaningful Space Situational Awareness (SSA) data and deriving actionable information from it is a continuous process involving state measurement, reconstruction, model estimation, forecasting/prediction, and reporting. As such, SSA is inherently a reactive process. With a slow (poorly performing) reactive SSA system, the belief or confidence possible in the predictive models it generates is abysmal. Data quantity, quality, and processing speed (including transmission and computation) fundamentally limit the latency with which a reactive SSA system performs. Minimizing this latency, resulting in a real-time reactive process, is the first step in achieving proactive SSA and enabling effective Space Traffic Management (STM).

Beyond the obvious need for volumes of high quality data provided in real-time, a second key factor in enabling an effective proactive-SSA system is the efficiency and accuracy with which predictive models can be computed from collected observation data. There are two important qualities of an optimally-performing predictive model generator. The first is the provision of immediate understanding of the first few evolutions of the domain state and does not cause temporal blind spots in the time it is taking to generate its predictions. For example, if it takes a week to generate a predictive model, then for that entire week, the model it is working on generating is useless. Even if the resulting model were perfect, that information arrives too late. The second, is the accuracy with which it captures natural evolutions of the domain state over increasing periods of time which enables increasingly longer duration propagation/prediction of state evolutions with errors below a desired tolerance.

Multiple phenomena are at odds with an optimal prediction system being correct for all time. I will name 3 here:

1. Unknown stochastic variations in modeled forces which result in additional structure in the error computed between the predicted model state and the truth.

2. Observed object intentional control of its own state. This can be spacecraft guidance, configuration changes, or attitude control which are unmodeled variations caused by the space objects' own behavior.

3. Natural unmodeled variations or events (e.g. collision, solar storms, spacecraft material aging, etc.) which happen on their own distinct time scales, are not part of the prediction model, and require discrimination from other variations.

Maintaining the predictive models of all objects in the space domain is the goal of an SSA system. Reporting changes and attributing cause, predicting effect, and assessing intent are all processes informed by good SSA. The accuracy of a predictive model directly affects follow-up observation processes, and all downstream products derived from them. Many of today's reactive SSA systems include sparse observation data, infrequent model updates, an exorbitant amount of time and resources spent reconstructing past behavior, and few actionable pieces of intelligence passed forward on relevant timelines to affect owner-operator decisions. This results in longer-duration blind spots and predictive models which are insufficient to inform effective and accurate follow-up observations. In contrast, a real-time networked systems approach enables proactive SSA and makes possible a host of courses of action left of an evolving danger to flight safety.

Given a set of observation data one wishes to derive an orbit from, there are many algorithmic options for determining the mapping from combinations of angles and range-range rate observations which enable the estimation of the 6 parameters necessary to describe the orbit from which the observations were taken. For any space object there is a characteristic time scale over which performing a batch fit to estimate a consistent trajectory for those observations makes sense. Every model being fit to a batch of observations contains implicit assumptions, and these assumptions only hold for a limited duration. In the case of active spacecraft, the assumption that the observed trajectory is ballistic, unforced motion has a maximum duration associated with the frequency with which that spacecraft performs its station-keeping maneuvers. Our data has shown that for many new spacecrafts in geosynchronous orbit this duration is shrinking, and with the adoption of more modern electric propulsion, the durations of non-ballistic behavior are getting longer. For debris, the duration for the validity of a ballistic fit is related to the models' ability to capture the objects interaction with non-conservative forces like solar radiation pressure. The rate at which the true behavior of the object and the fit model diverge is indicative of the magnitude of the mismodeling error. Without prior knowledge of a spacecraft's behavior or given insufficient knowledge in the form of un-modeled forces in the orbit estimation, fit residuals represent departures from the orbit fit over time. It is this fit residual structure which can allow inference that a given model is insufficient to represent the natural behavior of an object or distinguish between different behaviors. For these residuals to be informative, however, it is critically important to ensure that what remains within these errors is simply model error and not gross errors due to a failure to identify that the measurement set either was not of a single object (e.g. contained mis-tagged data) or observations taken during an event that violates the assumption of the underlying model (e.g. observations in the data set were taken during a maneuver). It is for these reasons that RANSAC combined with traditional initial orbit determination is investigated; to provide indicators of the fitness of a dataset for more detailed statistical orbit determination methods so that automated approaches to outlier rejection, maneuver detection, and behavioral analysis of space objects may be implemented.

This rest of this paper is organized into four sections. In section 3, I will introduce the Random Sampling Consensus algorithm and provide a comparison of batch least squares fits achieved for a simple linear regression, achieved with and without outliers, by using the algorithm's determination of a Consensus Set, e.g. the set of inlier measurements to a candidate model with maximum membership. In section 4, I will demonstrate embedding the Gooding angles-only initial orbit determination algorithm within a RANSAC process and show the benefit of selecting a RANSAC solution for IOD as opposed to just selecting three "well-spaced" measurements and applying the same algorithm. In section 5, I will introduce the connectivity matrix which describes the relationship between each measurement in a dataset and the set of models which meet a set of prescribed fit criteria and illustrated its behavior in cases where outlier behavior is present in the measurement set. Finally, in section 6, I will apply this approach to analyze data from two illustrative examples. One where a second model explainable by similar parameters is present in the data and is the source of the outliers, and one where the underlying model parameters change based on an action taken by the observed object.

3. RANDOM SAMPLING CONSENSUS

The Random Sampling Consensus (RANSAC) algorithm is a robust model fitting approach for estimation of model parameters given a set of data [1]. It is intended to be robust to the presence of outliers in measurements. The typical example of the simplest application of the RANSAC algorithm is the fitting of a line to 2D data. The implicit assumptions associated with processing a set of 2D points with a RANSAC line fitter are:

1.) There are enough meaningful inliers which map to a linear model such that a model can be determined and be best fit, and

2.) There is not a consistent set of outliers which fit a linear model and cause the algorithm to conclude that they are inliers.

The approach to applying the algorithm is to randomly select a minimal set of points, only two in the case of determining a line, and determine the parameters of the model (line). Consider a set of N = 100 measurements taken from the following observation model

$$X(t) = mt + b + \eta(0, \sigma) \tag{1}$$

Additionally, we will select 30 measurements and replace them with an outlier generation process defined by

$$Y(t) = X(t)) + \eta(0, \sigma_2) \tag{2}$$

Where $\sigma_2 >> \sigma$. For [i,j] = all N choose 2 combinations of inputs taken from the N = 100 data points where $i \neq j$ (a total of 4950), we may compute an estimate of the slope and intercept using:

$$\hat{m} = \frac{X(t_j) - X(t_i)}{t_j - t_i} \tag{3}$$

$$\hat{b} = X(t_j) - m(t_j) \tag{4}$$

Next, an error function which computes the distance of each data point from the determined model is computed, typically the orthogonal distance of the measurement and predicted measurement value based on the model.

$$\epsilon = \hat{X(t)} - X(t) \quad \text{where} \quad \hat{X(t)} = \hat{m}t + \hat{b} \tag{5}$$

Inliers are then determined by thresholding this distance. The model which fits the most inliers is selected after a maximum number of iterations. The value for the maximum number of iterations is either user selected or is bounded by the maximum number of combinations in the data set given the number of measurements and the minimum number of input measurements required to determine the parameters of the model being estimated. In the case of a line this is determined by n-choose-2; when we apply this technique with the Gooding IOD algorithm this is determined by n-choose-3. Fig. 1 plots the output of a simple observation process where measurements are generated from a linear model with Gaussian measurement noise and outliers generated by a corruption process. The resulting segmentation of a RANSAC line fitter is shown in the right-hand side of the same figure.

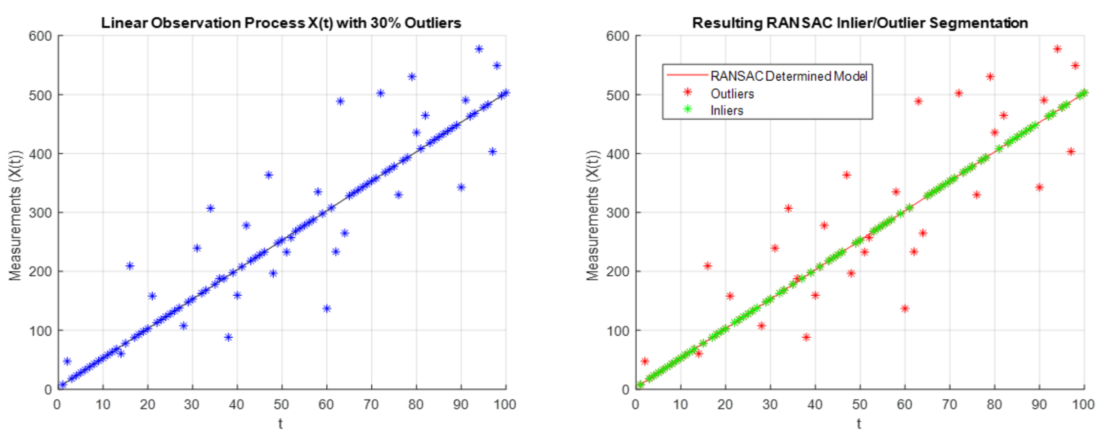

Fig. 1 Simple Linear Observer with outlier measurements and segmentation achieved using RANSAC parameter determination

Directly applying a batch least squares process to data with these characteristics can generate less than desirable results. In many areas where there is a visual front end, it is common to use RANSAC-based methods to facilitate effective pre-filtering of input data to ensure the more exquisite estimators are being applied to better behaved datasets. In the case of processing dense optical observations of space objects, methods like RANSAC can be very useful in facilitating the data-cleaning and data association steps involved in maintaining precise state knowledge on observed space objects. In the above simple linear example, the error in the estimate of the lines slope and intercept was significant in the presence of the outliers [4.9169, -44.7246] where the true values were [5,3]. Removing the outliers and applying a batch least squares estimate to the inliers returns [4.9978, 3.0248]. Avoiding these types of initialization errors in space object tracking is of great importance, which motivates the integration of this technique with angles only initial orbit determination.

4. RANSAC INITIAL ORBIT DETERMINATION

Extending this technique to determine the parameters of an orbit given angles-only data requires the RANSAC algorithm to be combined with an initial orbit determination routine of which there are many [2,3,4,5]. In this case three right ascension (RA) and declination (DEC) pairs are chosen as the minimal input set to exercise the Gooding angles-only initial orbit determination process [6] and obtain an estimate of an observed object's state vector at an associated time. The Gooding method takes in a sequence of 3 consecutive measurements of right ascension and declination (the spherical angles of a line of sight unit vector as references against the celestial background), their associated times, and estimates for the range values at t_1 and t_3. Internally to the algorithm, a Lambert Solver determines an orbit which is consistent with these parameters, and the algorithm solves for the set of parameters which minimizes the error over the measurements. It is known that as the subtended angle or arc length between these minimal input sets to a deterministic orbit estimator gets small, the reliability of the solution is low, due to many orbits being admissible to the fit given the noise in the measurements. With that said, perfect measurements - that is, any 3 samples taken from a known trajectory - will enable the correct determination of the orbit parameters invariant to the arc length between them. Extensions to the Gooding IOD technique have been developed [7] which address applying it to more than three input measurements, but still attempt to apply a batch error optimization over the input data as opposed to first determining the consensus set within the input data. For very precise measurements, the variation in the determined orbital parameters is quite small, but as the noise in the input measurements increases, the accuracy in the determined result can vary widely.

To illustrate this, we can investigate the simple two-body propagation of an object in Geosynchronous orbit (GEO) and compute the error in the computed state vector parameters using perfect measurements and measurements containing just 1 micro-radian of noise. Fig. 2 and Fig. 3 show perfect measurements and measurements with 1 micro-radian of error enabling Gooding estimates at every interior time point (not t = 0 or t = end) computed by passing the measurements from t_{k-1}:t_{k+1} into the IOD routine, respectively.

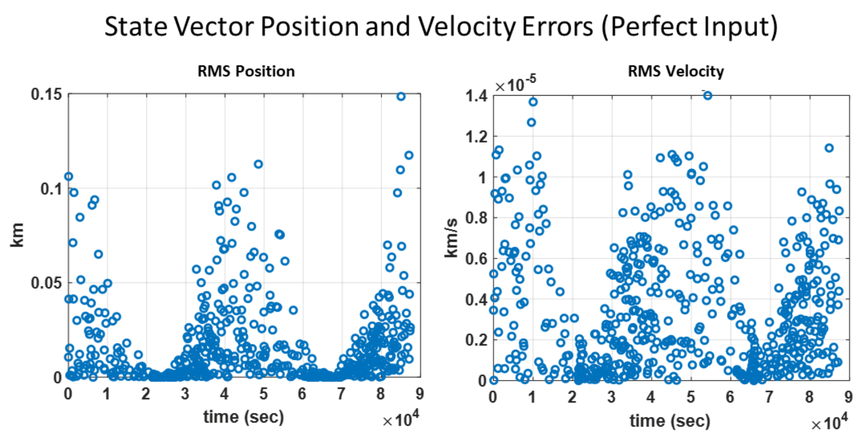

Fig. 2. IOD State Vector Errors on the order of meters and cm/sec or less using consecutive perfect measurements

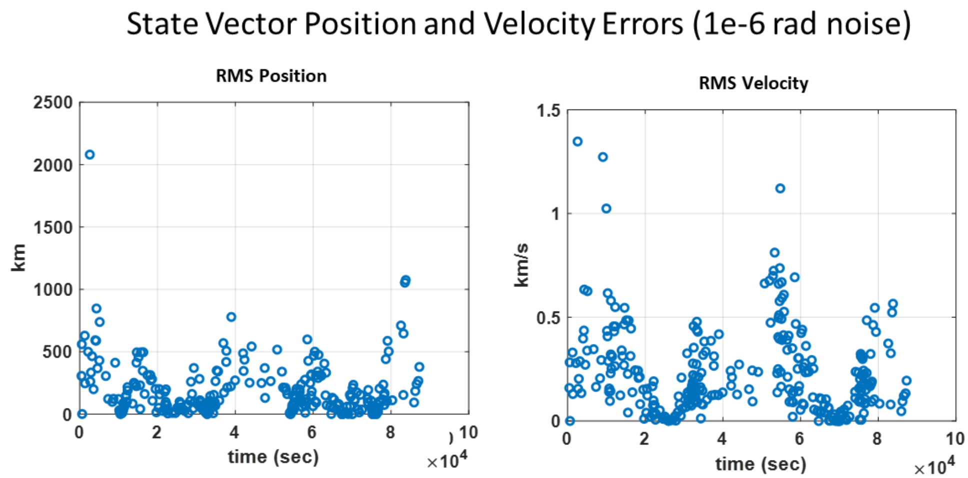

Fig. 3. IOD State Vector Errors rapid growth to 10’s to 100’s of kilometers and m/sec or using consecutive measurements with just 1 micro-radian of measurement noise

An excellent survey of IOD methods [8] including the Gooding method and their relative utility is recommended to the interested reader for a deeper understanding of the algorithm details. The determination of the state vector at this time, however, does not uniquely guarantee that the deterministic solution based on three given measurements is the best one. necessitating the use of a RANSAC solution. By applying a RANSAC approach, the Consensus Set within a candidate set of measurements is easily identified as the set of inliers to the model which contains the most inlier measurement. This ensures that a selected IOD solution is the best-performing IOD solution in the distribution shown in the figures above and avoids passing forward solutions which contain more extreme initialization error. Additionally, understanding the set of models which sufficiently contain a high fraction of inliers provides insight to the consistency of a set of measurements. Just like in the example of applying RANSAC to a linear observer with an outlier process resulted in the accurate segmentation of inliers and outliers, we can apply RANSAC with the Gooding IOD process integrated to a set of data provided by an optical system (or network of optical systems) which measure right ascension and declination. Outlier measurements can be preset in this data set for a host of reasons, including imagery where the detection of the space object overlapped with a bright star, additional objects present in collection data, wind buffeting, and other explainable sources for false positives or biased observations. Fig. 4 shows the DEC measurements of a simulated GEO with 30% of the data being outliers over the observation period just as in the linear example in section 3.

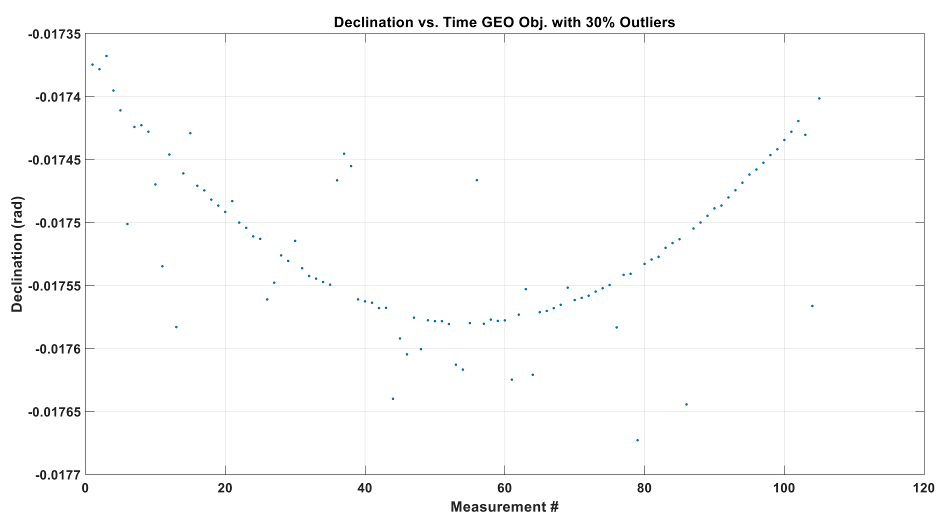

Fig. 4. Declination measurements of a GEO object with 30% outliers

The RANSAC tuning parameters need to be carefully chosen with an understanding of the measurement noise in the data, and the ability of an object to affect its orbit state (maneuver capability) as a function of time. The result is a robust capability to assess orbit models online with measurement data and flag changes in the tracking model. Follow-on precision OD methods would only be applied to inliers to a model given a set of parameters, thus decreasing the applications where precision OD techniques are applied to data where multiple models are required. To address this problem, the time sequence of RA and DEC measurements are grouped into time-ordered input triplets by enumerating the n-choose-3 set of index combinations. It is also well known that better IOD accuracies are typically obtained by selecting input triplets which are inliers to the underlying orbit, and subtend a larger arc as opposed to selecting three closely spaced measurements. For this reason, it is recommended to compute the total subtended time, index length, or change in the measurement value between these input triplets and sort the input sets by this value. This increases the likelihood of obtaining a better answer with fewer iterations, which is important for applications where there is insufficient time or computation to explore the full set of input options available. Applying RANSAC with Gooding IOD to the data in Fig. 4 results in the fit and segmentation shown in Fig. 5.

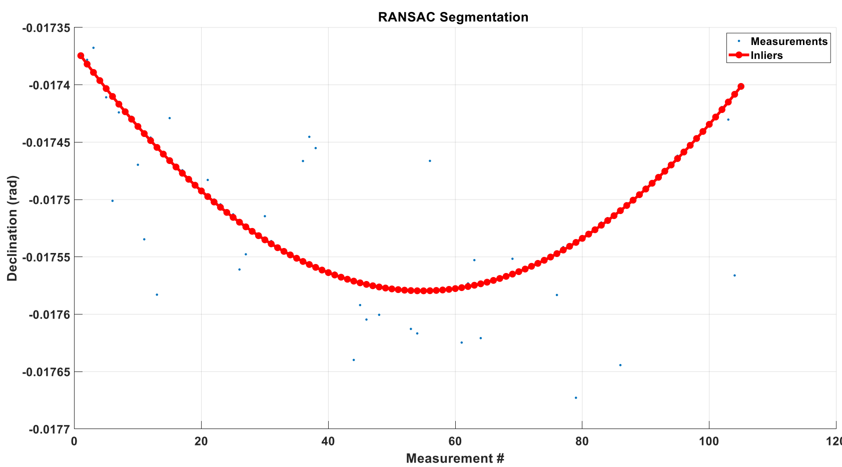

Fig. 5. Segmentation of Declination measurements of a GEO object with 30% outliers and resulting IOD fit from RANSAC with Gooding IOD

Enabling the reliable application of initial orbit determination techniques such as the Gooding method with the RANSAC approach has several benefits including the ability to reject outliers in input data, and to enable preliminary conclusions about data being considered for input to other more advanced processes. The next section illustrates the utility of analyzing more than just the single best fitting model generated by the RANSAC algorithm by considering the entire family of Gooding IOD output solutions which meet a set of fit criteria.

5. CONSENSUS SETS AND CONNECTIVITY MATRIX

Determining a set of RANSAC associated data points solves multiple problems at once. First, it provides a robust means to go from coarse initial estimates to more refined precision solutions by increasing the confidence that the points used are all inliers. Second, it has the potential ability to flag outliers which are potentially mis-tagged data, or which indicate the model changing within a given dataset, requiring the model fit to change as well. In the previous section, we applied the RANSAC approach along with the Gooding IOD method to accurately segment underlying inlier measurements which are all self-consistent with an underlying uncertain ballistic trajectory. As in other parameter estimation problems, the next step is to apply a batch least squares or other advanced parameter estimation method to the consensus set.

It is often overlooked however, that the best model output from a RANSAC process may not be unique. The RANSAC algorithm requires as input a threshold distance which determines whether a fit model and the measurements used to determine it are related as inliers or outliers. It also requires a threshold fraction of desired inliers; this parameter needs to be carefully selected with an understanding of the rate at which the source of the observation data produces outliers. It is almost always the case that the best-fitting model is not the only set of model parameters determined from the data that fit enough inlier measurements to the desired precision. By modifying the RANSAC approach to output every model that fits these parameters as opposed to just the model with the highest inlier membership within its consensus set, we can gain valuable insight into input datasets. While the human visual system is quite adept at recognizing patterns within measurement data, it is desirable to define algorithmic tools which validate our interpretation of datasets so that they may be used when automating tasks.

The connectivity matrix in this context is intended for just this purpose and is defined as an M by N matrix, where M is the number of measurement models determined by a RANSAC process which fit a set of specified criteria and N is the number of measurements in a dataset being analyzed. Fig. 6 shows the simplest case of a single miss-tagged outlier measurement present in observation data.

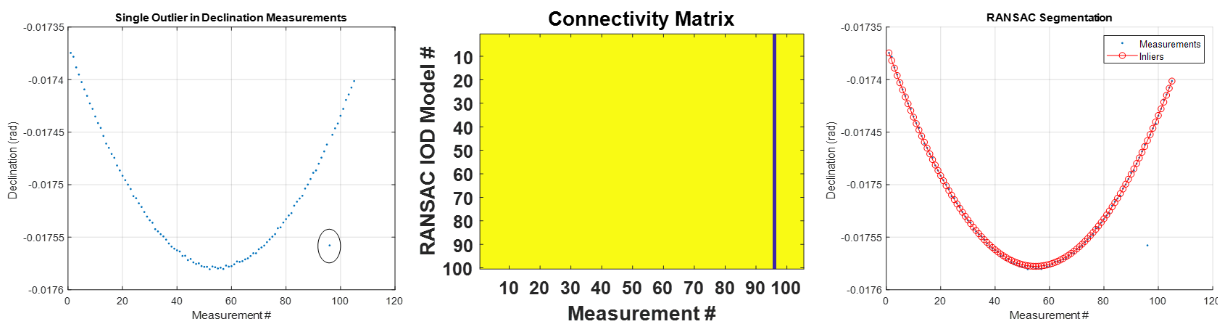

Fig. 6 A single outlier included in observation data, 100 models meeting fit criteria, Connectivity Matrix and Resulting Segmentation

This mis-tagged measurement fails to be associated by any of the models which meet the RANSAC criteria specified, which is strong evidence that the measurement should be removed from any downstream analyses for the object associated with the rest of the measurements. There are several aspects of this matrix which indicate how well behaved a data set is and how well it interacts with the models being fit to it. First, the number of the n-choose-k combinations of models that can be determined which fit the desired fraction of the data to within the measurement noise provides an indicator as to what degree the data being processed conforms to the underlying assumptions of the models. Second, the number of measurements which are members of the consensus sets of these models can explain if one model is enough to explain all the measurements or if more than one model is needed. We will see two examples of this in the next section. Summing the matrix along the rows will explain the number of models to which each measurement is an inlier and summing the matrix along the columns will compute the membership of the consensus set for each model. Strong outlier measurements appear as columns of zeros and may easily be identified and removed using indexing based on the matrix. Contiguous groups of outlier columns are good indicators that the underlying model of the observed object has changed: in other words, a subset of the data does not fit the same model. It is also possible to segment the inliers and outliers and re-run RANSAC to determine if there are any other models which may explain structure underlying the non-members of the consensus set that best explains the larger dataset. Fig. 7 shows the connectivity matrix for the 30% outlier case analyzed in the previous section.

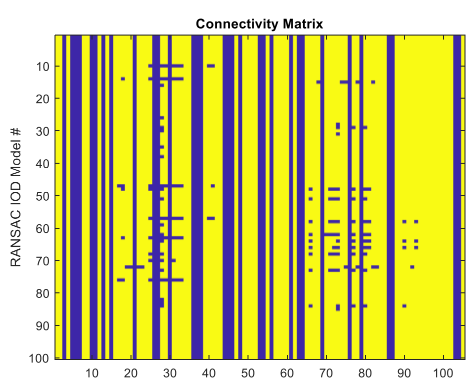

Fig. 7 Connectivity Matrix for 30% outlier rate example

Notice there are several clear outlier columns in this matrix. It is not clear from this data structure however, if there is any correlation between the outliers. At this point, segmenting the consensus set and computing a least-squares fit to the inliers will result in an accurate determination of the orbit for the observed object. Subsequent attempts to fit the outliers will not work, so they will not be demonstrated here. In the next section we will investigate a case where the outliers are indeed correlated to an orbit.

6. ILLUSTRATIVE EXAMPLES

In this section we consider a few demonstrative examples illustrate the potential for methods based on RANSAC to inform an analyst of subtle challenges which may go undetected in the direct application of batch or sequential estimation methods and show up as structure in residuals. These challenges include violations of assumptions that the data being processed represent a single object, moving ballistically, and are measured by a network of observers whose measurements can be expected to be zero-mean Gaussian distributed around the desired motion model to be determined.

Detecting Non-Ballistic Behavior in Measurement Sets:

Fitting a single trajectory to a set of measurements implicitly assumes that there have been no departures from the orbit model over the time of the measurement duration used for the fit. This presents a dilemma. To increase the probability that no maneuvers might have occurred, it is necessary to decrease the time duration of the data used for the fit. For an accurate orbit estimate, or representation of a nominal trajectory, the longer the duration where the orbit can be assumed to be constant, the better. Given that these two criteria are at odds, methods which can determine if an increasing amount of associated data on an object can continue to be assumed to be consistent or not are valuable. To demonstrate how the application of RANSAC and Gooding IOD can be used to do this we will apply the techniques to a dataset generated by two separate observed orbits, switching between them at a specific time in the measurements. This is intended to simulate the case where the underlying model parameters change due to an action taken by the observed object. Fig. 8 shows the input data.

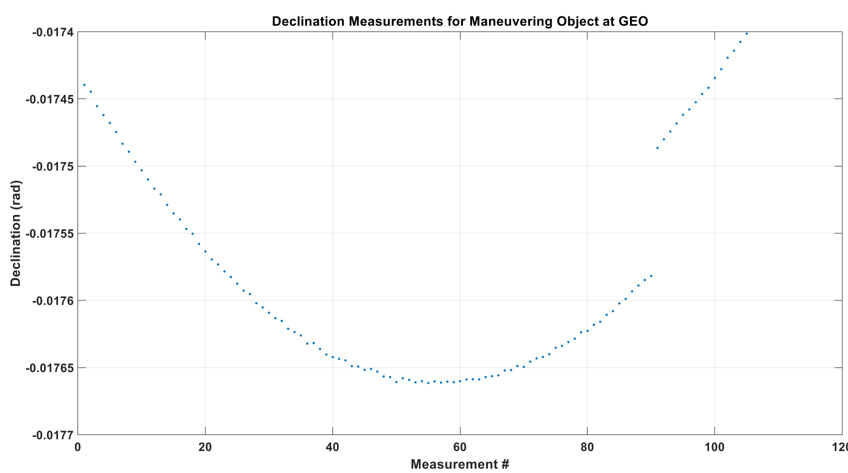

Fig. 8 simulated measurements of a maneuvering deep space object

In this case the underlying orbit model is changed at measurement 91. Depending on the measurement frequency, this may be a realistic representation, or evidence of a maneuver may be present over several measurements. Either way, this input data set illustrates the utility of analyzing the connectivity matrix generated when applying a RANSAC Gooding IOD process. Fig. 9 shows the resulting segmentation achieved by RANSAC Gooding IOD. In this case the consensus set of the best fitting model fits the first 90 measurements well but cannot be relied upon to predict the future state of the observed object.

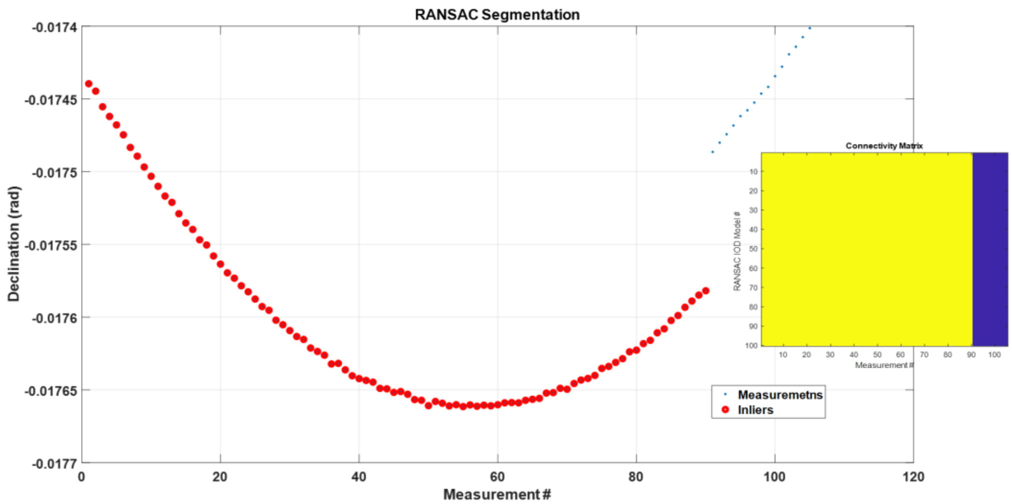

Fig. 9 RANSAC Segmentation achieved for object with changing model parameters

It is clear from the connectivity matrix that a new model is needed and must be fit to the last 15 measurements. This is useful information. Obtaining a batch least squares fit and learning later that it is diverging faster than expected can have serious consequences. Knowing at this level of detail that the model parameters may be changing can be useful in directing actions through sensor tasking.

<u>Detecting a nearby object:</u>

Follow-up analysis on outlier measurements can result in the detection of a second object close to the object being analyzed. Instead of simulating a change in model parameters, a second object is simulated and used as the source for outlier measurements. Given that probability of detection of any object is not always 1, it is possible that two closely-spaced dim objects may only be observed every few frames, making for interesting combinations of object observations at each time step. Two closely-spaced measurements within the same focal plane during the same exposure is strong evidence that two objects are present, but as false alarm rates and outlier measurements increase, this is not always an obvious conclusion. To illustrate using this process to identify the presence of multiple objects in a set of observations, we will again apply the RANSAC Gooding IOD process to the data shown in Fig. 10.

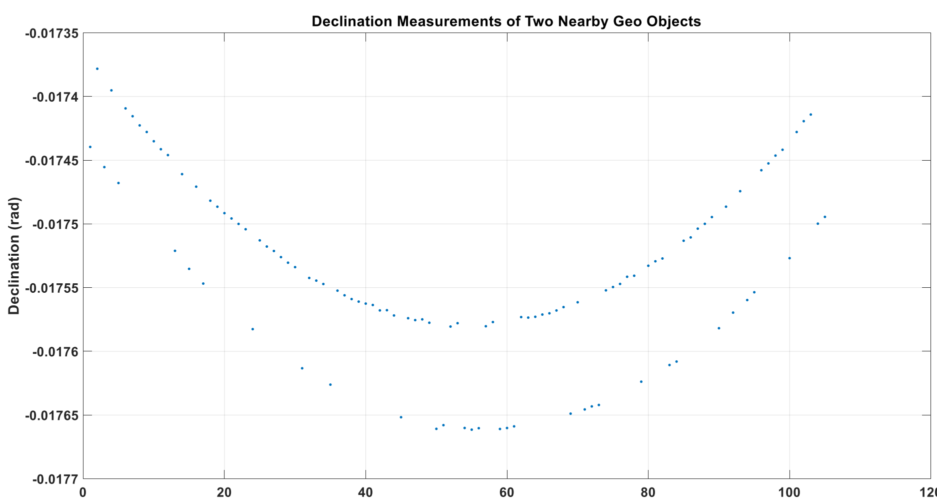

Fig. 10 input measurement data sampled from two objects

Again, it is obvious by inspection that there is structure to the outlier measurements from the desired consensus set. It is therefore not surprising at this point that the RANSAC Gooding IOD algorithm achieves the segmentation shown in Fig. 11.

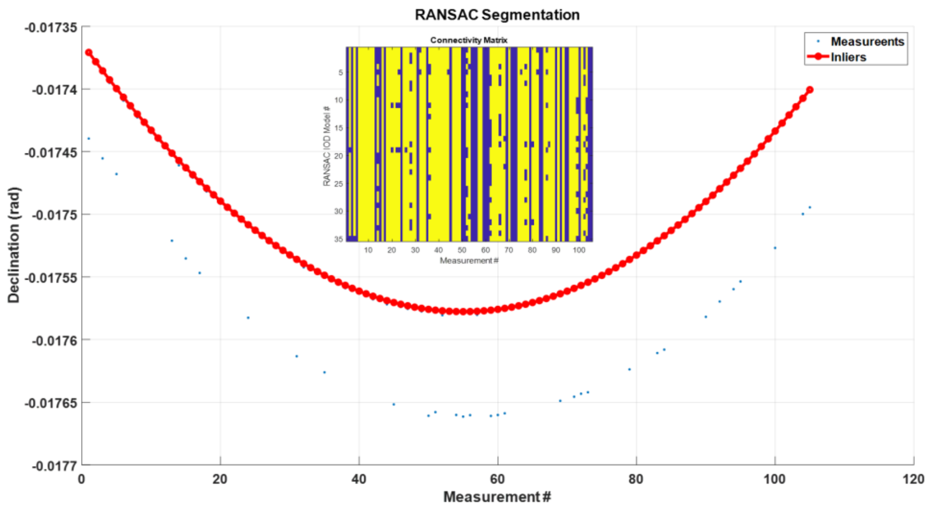

Fig. 11. RANSAC Gooding IOD Segmentation of multi-object input data and connectivity matrix

The connectivity matrix in this example looks very similar to the one observed in the 30% outlier case in section 5. The connectivity matrix does not necessarily indicate correlation in outliers, though when they occur in contiguous groups like in the maneuver example, it can be concluded that they are outliers for a common reason. To determine the presence of the second object, the outliers must be grouped together and a RANSAC Gooding IOD process can again be applied. This results in the following model being determined in Fig 12.

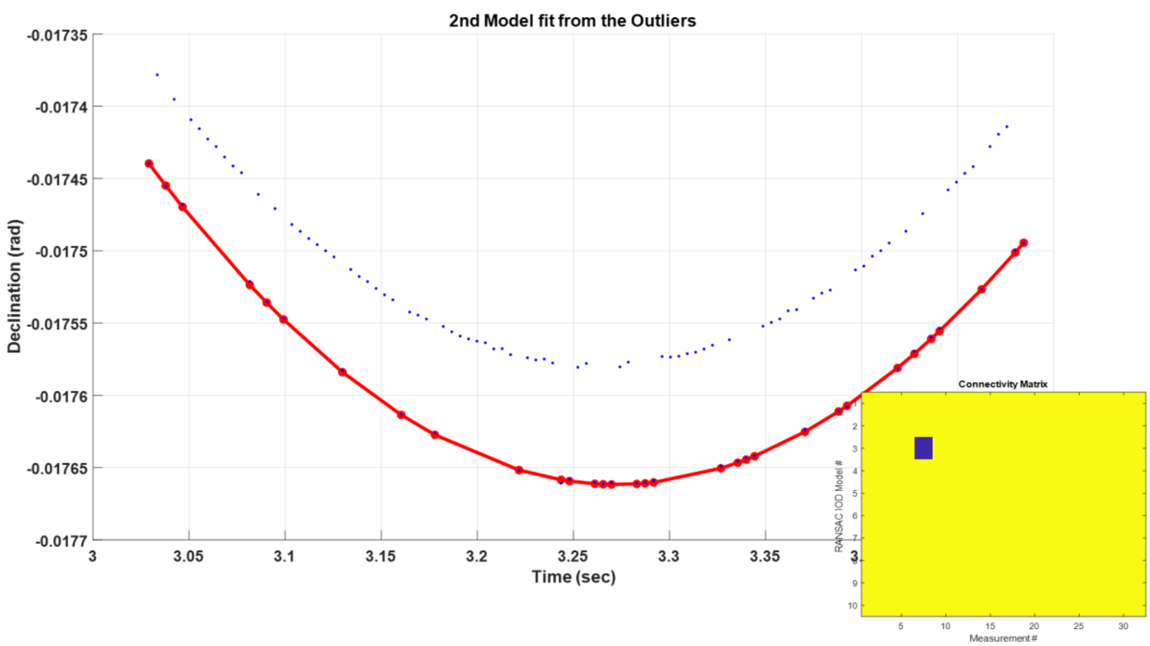

Fig. 12. Determination of 2nd model present in the outliers of the consensus set

Notice a fit is determined which considers all these measurements as inliers, as is further supported by the connectivity matrix being nearly all ones. This is a strong indicator that the outliers are in fact correlated and require an additional orbit to explain them. A least squares orbit fit of this data will be effective in predicting future motion of this second object, as there are no obvious outliers within the data which may skew the resulting estimate.

7. CONCLUSIONS

This work applied the Random Sampling Consensus algorithm to perform multiple data analysis functions, which would typically be upstream from more advanced statistical orbit determination functions. By incorporating the Gooding angles-only initial orbit determination algorithm as the parameter estimator within the RANSAC process, it was demonstrated that reliable state vector estimates from a sequence of noisy observation data could be easily obtained. This process was effective in the presence of a high frequency and volume of outlier measurements, whether uncorrelated or correlated. For the case of correlated outlier measurements, the process was demonstrated in the presence of outliers either originating from a second object or from a change in underlying orbit parameters due to a maneuver during data collection. The extension of the RANSAC process to provide not only the model with the highest number of inliers, but all models which meet a set of fit criteria, was shown to provide insight as to the self-consistency of the data collected within the parameter space being determined. In this case, the family of models which meet a set of criteria inform the confidence with which it can be concluded that the data represent observations of a single ballistic object by a network of sensors exhibiting well-behaved Gaussian noise. The connectivity matrix which relates the inlier and outlier measurements to the family of determined models provides utility in defining logical indexing for the removal of outlier measurements, determination of additional structure in outlier data, and the confidence with which conclusions from follow-on estimation algorithms can be used in a decision-making process.

All examples in this work are constructed to illustrate the ideal application of this approach to collected data and as such avoid several pitfalls which need to be addressed when applying the approach to real data. The agreement of the orbital parameters between the two-body propagated states and the results of the Gooding IOD process are optimistic. It is well known that more fidelity is needed in the force model to accurately represent the motion of space objects. This is important since there is a duration over which IOD estimates of this type can provide reliable initial guesses within the domain of convergence for more advanced estimators. This duration is directly associated with the model error growth rate which can be evaluated by propagating a state vector with a more advanced force model and computing the error between it and the two-body assumption. As space object maneuvers are occurring more frequently today, and the opportunity to observe objects more frequently is becoming ubiquitous, there are additional ways to manage this error growth with more frequent sensor updates.

Future work will consider the direct inclusion of more advanced estimation routines within a RANSAC process, as well as applying the RANSAC Gooding IOD process to observations of real objects to assess the timelines over which the approach described here can be considered robust and reliable. Additionally, the processing of RA and DEC measurements alone can provide some insight to what was observed and the underlying orbit parameters which describe the trajectories of the observed objects. However, when the additional context of optical imagery and photometric values associated with each detection are available, other characterization algorithms may be used to identify data which may violate the implicit assumptions of down-stream parameter estimation routines, and to identify objects which may require analyst attention. Additional future work is also focused on the fusion of photometric indicators to develop autonomous processes to identify when interesting events are observed, so that they are not simply passed to downstream processing without alerting an analyst to indicators of possible un-modeled or unexpected behaviors in collected data. The goal is to enable proactive, rather than simply reactive, SSA analysis.

8. REFERENCES

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