Numerical conservation of exact and approximate first post-Newtonian energy integrals

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Abstract

The incorporation of Albert Einstein's theory of gravitation is necessary for any satellite mission requiring highly precise orbit information. For near-Earth objects, this is achieved using the first post-Newtonian approximation. In this article, we derive an exact analytical expression for the energy of a test particle in the spherically symmetric Schwarzschild field of the Earth by seeking a Jacobi-like integral associated with the post-Newtonian equations of motion. The energy integral so obtained contains exponential terms which when approximated produce results which are well established in the literature. Using structure-preserving symplectic numerical integration schemes, we demonstrate that the integral containing exponential terms more accurately describes the true conserved dynamics of test particles in the PN regime as compared with the existing approximations of the energy invariants.

1 Introduction

The answer to the long-standing, early 20th century anomaly associated with the precession of Mercury's perihelion was provided by Albert Einstein's highly non-linear and covariant theory of gravitation; General Relativity (GR) [8, 17]. The classical, instantaneously acting inverse square law of Newtonian gravity was superseded by GR, where gravity is described as a consequence of spacetime curvature with finite propagation speed as provided by the postulates of special relativity. The field equation for Newtonian gravity for an attracting body of density ρ is given by the well known Poisson equation

$$\nabla^2 U = -4\pi G\rho, \tag{1}$$

where the gravitational potential and Newton's gravitational constant are given by U and G respectfully. However, unlike the linear field equations of Newtonian gravity, the Einstein field equations of gravitation constitute a system of ten non-linear, coupled, second order partial differential equations and are given by [17]

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2}$$

where the speed of light, the Einstein and energy-momentum tensors are given by c, $G_{\mu\nu}$ and $T_{\mu\nu}$ respectfully [8, 17]. Due to the complicated structure of the Einstein field equations, existing exact analytical solutions are limited and involve symmetry and dynamical constraints in order to simplify the field equations. Although inherently different theories of gravitation; some important similarities exist. As discussed in [6], an important similarity arises when discussing the motion of bodies that produce weak gravitational fields and move slowly as compared with the speed of light. In such regimes, solutions to the Einstein field equations can be obtained by approximate methods.

The incorporation of GR for orbit determination of near-Earth objects, geodetic techniques [9], calculation of solar system planetary ephemerides and deep space navigation is achieved using the first post-Newtonian (PN) approximation [12]. The recommended PN equations of motion for near-Earth objects are maintained and published by the International Earth rotation and Reference Systems service (IERS) [11]. The PN equations of motion account for Schwarzschild, Lense-Thirring and geodetic accelerations which arise due to the relativistic contribution of the spherically symmetric gravitational field of the Earth, the rotation of the Earth (the so-called frame-dragging effect [1]), and the relativistic contribution from the gravitational field of the Sun respectfully (see [9, 18] for detailed calculations and corresponding magnitudes of Schwarzschild, Lense-Thirring and geodetic accelerations applied to GNSS and geodetic satellites).

In a recent article [10], the present authors derive a new energy integral associated with a test particle in the first PN regime. The integral so obtained involves an exponential function when suitably approximated produces well established results. Since both exponential and approximated terms behave quite differently mathematically, the question arises as to whether or not there are situations when retaining the full exponential term predicts quite different behaviour to the approximating series. The present paper considers the long term conservation of the aforementioned energy integral when compared with the corresponding Taylor series approximated expressions given by Eqs. (14) and (15) respectfully. We show that the magnitude of drift in energy during numerical simulations is greater for the approximated invariants which suggests that explicitly retaining the exponential functions is more advantageous for the long-term description of PN dynamics. The paper is organised as follows: in Sec. (2) and (3) we present a brief derivation of the PN equations of motion and energy integrals respectfully. Sec. (4) introduces symplectic integration schemes and preliminary results regarding the long-term conservation of the previously mentioned energy integrals. Finally, the results are discussed in Sec. (5).

2 Equations of motion

In this section we provide the relevant mathematical tools necessary to derive the equations of motion associated with a test particle in the first PN regime. There exists several equivalent PN frameworks where it should be noted that we adopt the approach provided by the Damour, Soffel, Xu (DSX) formalism [2] (cf. [7] and references therein for a thorough review). Hence, the metric tensor components are given by

$$\begin{aligned} g_{00} &= e^{-2W/c^2} = 1 - 2W/c^2 + 2W^2/c^4 + O(c^{-6}), \\ g_{0j} &= 4W_j/c^3 + O(c^{-5}), \\ g_{ij} &= -\delta_{ij} e^{-2W/c^2} = -\delta_{ij}(1 + 2W/c^2) + O(c^{-4}), \end{aligned} \tag{3}$$

where δ_{ij} denotes the usual Kronecker delta function and scalar and vector gravitational potentials are given by W and W_j respectfully. We note that the gravitational field is completely described by the four-potential W^μ given by $W^\mu = (W, W_j)$ where the space-space and time-time components of the gravitational field are described by the generalised post-Newtonian scalar potential W and the time-space components are described by the *gravitomagnetic* vector potential W_j which accounts for rotating or moving masses [7, 15]. The convention adopted is Latin indices account for spatial variables ($j = 1, 2, 3$), while Greek indices account for space-time variables ($\alpha = 0, 1, 2, 3$). The 0th component $X^0 = cT$ is reserved for coordinate time. Finally, we assume that the potentials W and W_j

are spatially dependent such that $W(X^j)$ where $X^j = (X^1, X^2, X^3)$ are the usual geocentric Cartesian coordinates associated with the *local* reference system discussed in the DSX formalism.

The motion of a test particle is described by the geodesic equation of motion given by [17]

$$\frac{d^2X^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dX^\beta}{d\lambda}\frac{dX^\gamma}{d\lambda} = 0, \tag{4}$$

where the Christoffel symbols of the second kind are given by $\Gamma^\alpha_{\beta\gamma}$ and λ is used to parametrise the worldline of the test particle [17]. Using the 0th component of the geodesic equation, we can express the motion of a test particle by a *modified* geodesic equation which is now parametrised in terms of coordinate time given by [7, 12]

$$\frac{d^2X^i}{dT^2} = -c^2\left(\Gamma^i_{00} + 2\Gamma^i_{0j}\frac{V^j}{c} + \Gamma^i_{jk}\frac{V^j}{c}\frac{V^k}{c} - \left[\Gamma^0_{00} + 2\Gamma^0_{0j}\frac{V^j}{c} + \Gamma^0_{jk}\frac{V^j}{c}\frac{V^k}{c}\right]\frac{V^i}{c}\right), \tag{5}$$

where we have introduced $V^j = dX^j/dT$ as coordinate velocity of the test particle. The explicit calculations of the Christoffel symbols for potentials W and W_j are given in [12, 16, 17]. Hence, the final form of the equations of motion for a test particle are given by

$$\frac{d^2X^i}{dT^2} = \partial_i W + \frac{1}{c^2}\left[\left(V^2 - 4W\right)\partial_i W - 4V^iV^k\partial_k W - 4(\partial_i W_k - \partial_k W_i)V^k\right], \tag{6}$$

where $\partial_i = \partial/\partial X^i$ is the usual gradient operator and $V^2 \equiv V_jV^j$. The simplification of Eq. (6) is greatly facilitated by assuming the test particle orbits a non-rotating ($W_j = 0$), isolated, and spherically symmetric Earth with scalar potential given by $W = GM/R$ where $R = |X^j|$. The assumption $W_j = 0$ is justified in the context of Maxwellian electromagnetism, where the vector potential W_j produces a Lorentz-like force which does not contribute to the energy of the test particle [10]. This is shown explicitly in [10] through algebraic manipulation of (6). Hence, the PN equations of motion for a test particle in the spherically symmetric gravitational field of the Earth are given by

$$\frac{d^2X^i}{dT^2} = -\frac{GM}{R^3}X^i + \frac{1}{c^2}\left[\left(4\frac{GM}{R} - V^2\right)\frac{GM}{R^3}X^i + 4\frac{GM}{R^3}\left(X_jV^j\right)V^i\right]. \tag{7}$$

3 Energy integrals

The conservation laws associated with energy, angular momentum and linear momentum give rise to invariant quantities known as first integrals. Newton's second law of motion is derivable from the gradient of a spatially dependent potential. Due to this property and following multiplication by the velocity components dx^i/dt, the equations of motion are immediately integrable, giving rise to an invariant quantity often referred to as Jacobi's integral [4]. A corresponding first integral associated with (6) exists and is deduced in [10] for *arbitrary* scalar and vector potentials W and W_j respectfully. In this section, we derive the energy integral associated with a test particle in the Schwarzschild field of an isolated, non-rotating Earth.

The scalar product between (7) and velocity components V_i yields

$$\frac{d}{dT}\left(\frac{V^2}{2} - \frac{GM}{R} + 2\left(\frac{GM}{cR}\right)^2\right) = -3\left(\frac{V}{c}\right)^2\frac{d}{dT}\left(\frac{GM}{R}\right), \tag{8}$$

where we have used the following identities

$$-\frac{GM}{R^3}\left(V_iX^i\right) = \frac{d}{dT}\left(\frac{GM}{R}\right), \qquad 4\left(\frac{GM}{cR^2}\right)^2\left(V_iX^i\right) = -2\frac{d}{dT}\left(\frac{GM}{cR}\right)^2. \tag{9}$$

Eq. (8) constitutes a simple, first order, linear, ordinary differential equation for V^2 as a function of $W = GM/R$ with corresponding solution given by

$$V^2 = \frac{5c^2}{9} - \frac{4}{3}\left(\frac{GM}{R}\right) + Ce^{-6GM/(c^2R)}, \tag{10}$$

where C is an arbitrary constant of integration. The identification of C is achieved in two equivalent ways which will be used for comparison in Sec. (4).

We observe the Newtonian gravitational potential is prescribed modulo an arbitrary constant. Hence, Eq. (10) is equivalently expressed as

$$V^2 = C_1 - \frac{4}{3}\left(\frac{GM}{R}\right) + C_2 e^{-6GM/(c^2R)}. \tag{11}$$

Further, if we impose the condition that Eq. (11) tends to the Jacobi integral $V^2 = \text{const.} + 2W$ in the limit $W \to 0$, we deduce $C_2 = -5c^2/9$. Hence, Eq. (11) is given by

$$V^2 = C_1 - \frac{4}{3}\left(\frac{GM}{R}\right) - \frac{5c^2}{9} e^{-6GM/(c^2R)}. \tag{12}$$

Alternatively, the value of C in (10) can be deduced using the definition of energy at infinity, where in the limit as $R \to \infty$, the velocity $V^2 \to 2\mathcal{E}$ where $\mathcal{E}$ is the Newtonian energy associated with a test particle at spatial infinity. Hence, Eq. (10) now reads

$$V^2 = \frac{5c^2}{9}\left(1 - e^{-6GM/(c^2R)}\right) - \frac{4}{3}\left(\frac{GM}{R}\right) + 2\mathcal{E}e^{-6GM/(c^2R)}. \tag{13}$$

We note that the equivalence of both approaches for (12) and (13) is demonstrated explicitly in [10]. Expanding the exponential functions in (12) and (13) to PN order yields

$$V^2 = C_1 - \frac{5c^2}{9} + 2\left(\frac{GM}{R}\right) - 10\left(\frac{GM}{cR}\right)^2, \tag{14}$$

and

$$V^2 = 2\left(\frac{GM}{R}\right) - 10\left(\frac{GM}{cR}\right)^2 + 2\mathcal{E}\left(1 - 6\frac{GM}{c^2R}\right), \tag{15}$$

respectfully. We note Eq. (15) is well established and is discussed in [12, 14]. We have derived two invariant quantities of exponential order describing the energy of a test particle orbiting a spherically symmetric Earth given by Eqs. (12) and (13) with the corresponding Taylor approximations given by Eqs. (14) and (15). The exponential representation was derived for arbitrary scalar gravitational potential in [10] and does not appear in popular PN literature [12, 14]. The literature assumes the Taylor series approximations are sufficient which gives rise to the important question as to whether there is any benefit in retaining the exponential functions in describing the dynamics associated with bodies in the PN regime. The answer is in the affirmative and is demonstrated in the following section.

4 Numerical methods

In general, classical numerical methods for solving differential equations such as the well-known 4th order Runge-Kutta scheme do not preserve first integrals such as energy and angular momentum. This leads to a dissipative-like effect giving incorrect qualitative behaviour over long integration periods. Symplectic integration schemes [5, 13] have been successfully developed to combat such dissipative effects associated with classical schemes providing almost near conservation of energy integrals. Applications of symplectic integration schemes arise in areas such as

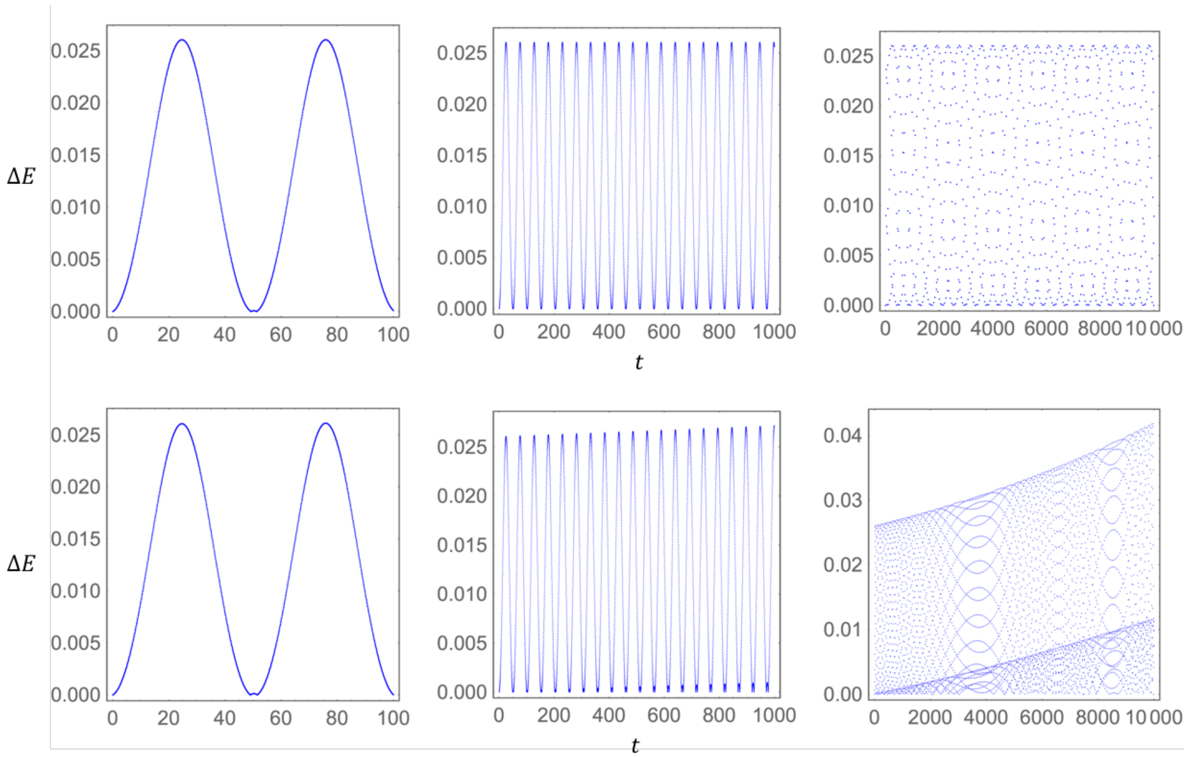

Figure 1: Long term conservation of exponential energy integral (12). **Top row:** Bounded energy associated with second-order Gauss Legendre symplectic integrator. **Bottom row:** Non-linear secular drift in energy associated with 4th order Runge-Kutta scheme.

molecular dynamics, particle accelerator physics and celestial mechanics (see [5, 13] and references therein). Symplectic integrators are aptly named due to their ability to preserve the symplectic structure associated with Hamiltonian systems [3, 13] and provide an ample opportunity to compare the long term evolution of the invariant quantities (12) and (13) with their associated approximations (14) and (15) respectfully. Fig. (1) demonstrates the advantages of using symplectic integration schemes by comparing the long term drift in energy using a second-order Gauss-Legendre scheme (symplectic) with a 4th-order explicit Runge-Kutta. It is shown that the long term error in energy ΔE associated with (12) remains bounded and close to the true value when using symplectic schemes and shows non-linear secular drifts when using a 4th-order Runge-Kutta.

Introducing dimensionless variables according to $x^j = X^j/X_c$ and $t = T/T_c$ where X_c, T_c are characteristic units given by $X_c = GM/c^2$ and $T_c = GM/c^3$ respectfully greatly simplifies Eq. (7) computationally. Hence, the non-dimensionalised equations of motion of a test particle in the Schwarzschild field of the Earth are given by

$$\frac{d^2x^i}{dt^2} = -\frac{x^i}{r^3} + 4\frac{x^i}{r^4} - v^2\frac{x^i}{r^3} + 4\frac{v^i}{r^3}\left(x_j v^j\right). \tag{16}$$

To demonstrate a distinct difference between retaining the exponential functions in (12) and (13) over the corresponding Taylor series approximations (14) and (15) we perform several simulations by numerically integrating the non-dimensionalised PN equations of motion (16) using a second-order Gauss-Legendre integration scheme. We note that all simulations are subject to initial conditions given by $x^i(0) = (3.6, 2.3, 1.1)$, $\;v^i(0) = (-0.004, -0.003, 0.0002)$ and integration time step $h = 0.01$. The initial conditions are obtained from NASA's Jet Propulsion Laboratory

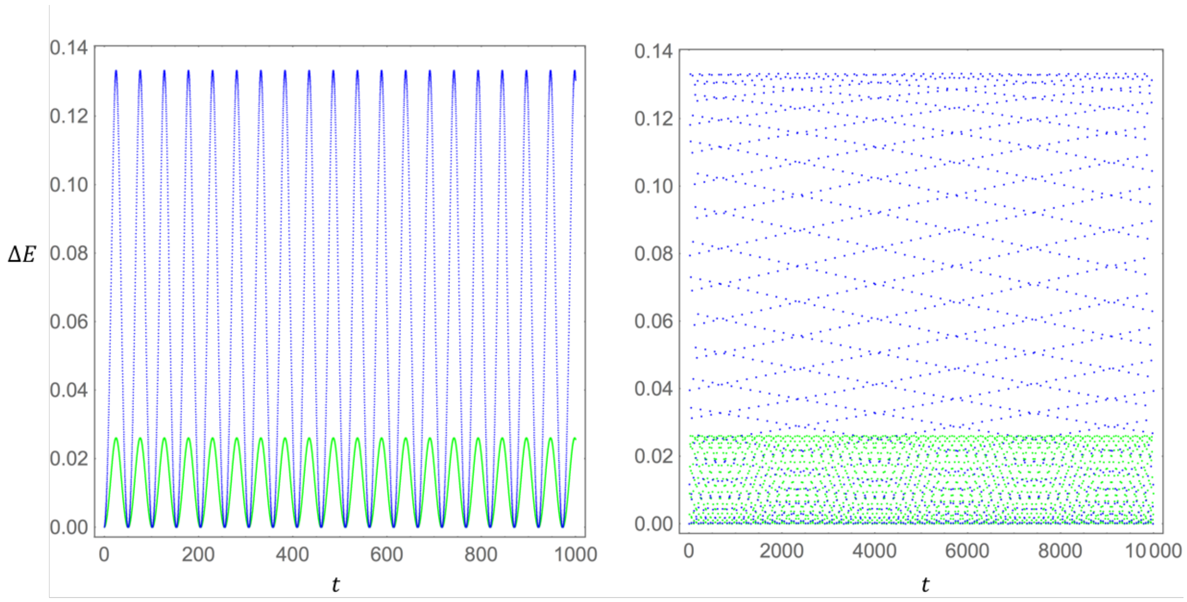

Figure 2: Error in energy integrals (12) (green) and (14) (blue). Errors associated with (12) and (14) are bounded due to nature of symplectic schemes. Magnitude of error is larger for Taylor series approximation.

(https://ssd.jpl.nasa.gov/) for the LAGEOS-1 spacecraft which have been appropriately scaled for the simulation. Fig. (2) and (3) demonstrate the error in magnitude is greater for both approximation methods (14) and (15) indicating that exponential representation of energy integrals given by (12) and (13) more closely resemble the conserved dynamics associated with a test particle in the PN regime over long term integration. Further, Fig. (3) indicates that the error in (13) is almost completely conserved with an error of the order 10^{-10}.

5 Conclusion

The PN literature assumes the energy associated with a test particle is sufficiently represented by approximate forms given by Eqs. (14) and (15). By seeking an exact Jacobi-like integral in [10] and in Sec. (3); an exponential term arises in the energy integral associated with a test particle. In this article we have shown that the exponential term in the energy integrals (12) and (13) more closely resembles the true conserved dynamics of test particles in the PN regime. The long term magnitude in error for the Taylor series approximations (14) and (15) is shown to be larger in both cases. Further, the long term behaviour in error associated with Eq. (13) is shown to be of order 10^{-10}. It can be shown that the energy integral given by Eq. (13) is completely conserved (up to round-off error) by increasing the order of the symplectic scheme and is discussed in Fig. (4) . The complete conservation of (13) requires further investigation.

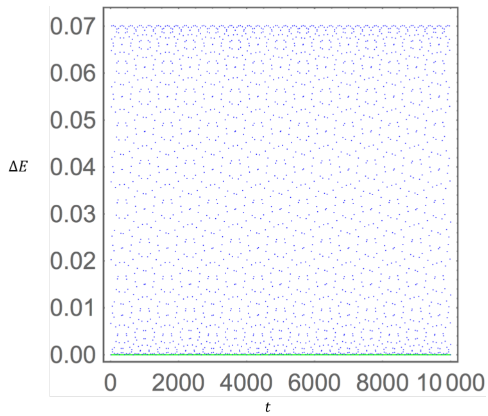

Figure 3: Error in energy integrals (13) (green) and (15) (blue). Error for both energy integrals is bounded. Magnitude of error associated with (13) is of order 10^{-10}.

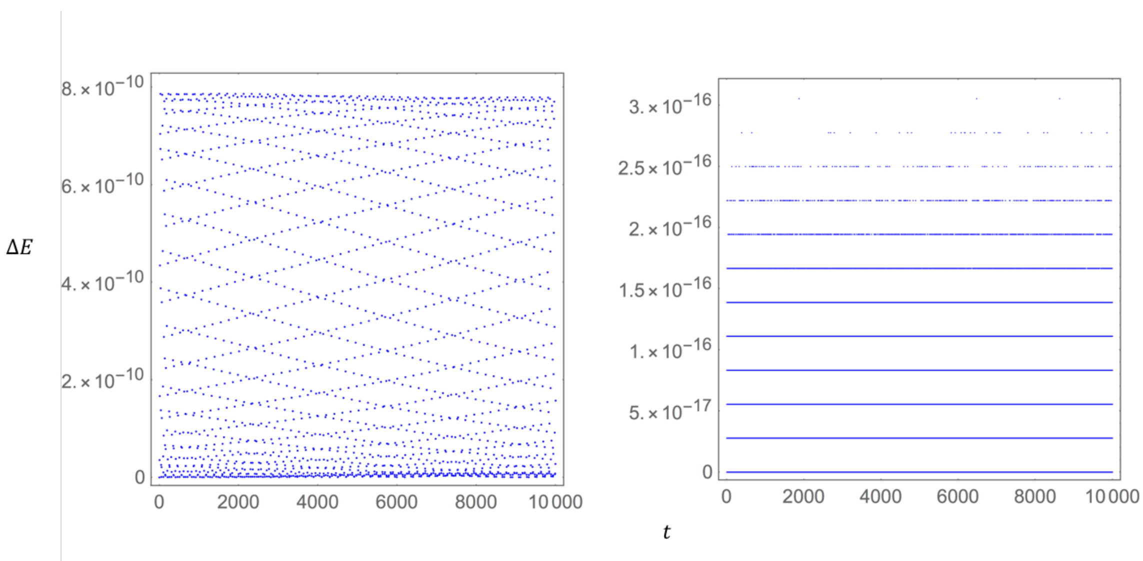

Figure 4: Long term conservation of exponential representation of energy integral (13). **Left:** Bounded error associated with Eq. (13). **Right:** Complete conservation (up to round-off error) of Eq. (13) using a symplectic 10th order implicit Runge-Kutta scheme.

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