Risk-Based Decision-Making for Space Traffic Management

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ABSTRACT

Each space conjunction event is unique. This uniqueness arises not just from the situational specifics pertaining to the conjunction at hand, such as the number and maneuvering abilities of the relevant objects, but also the nature of the available data: The quality and completeness of the data contribute to the uniqueness of each event. Current practice also relies on employing a single, pre-established probability of collision threshold that is applied regardless of the unique realities of each individual conjunction. In place of these current practices, we propose a formal, subjective risk/benefit analysis for conjunction assessment and decision-making. Such an analysis replaces the use of a pre-determined probability of collision threshold with a rational set of criteria that are optimized around the unique aspects of the situation at hand. The proposed analysis also naturally provides a prescription of how best to maneuver given the uniqueness and complexity of the conjunction event.

The proposed risk-based analysis can accommodate highly complex decision environments as well as the unique aspects of the relevant data including but not limited to: uncertain but known multi-object locations; uncertainty associated with the number and locations of multiple conjuncting objects; and incomplete or altogether missing data (as in the case of known lost objects in the vicinity of a conjunction event). Specifically, in this paper we draw from various mathematical frameworks, such as multi-target, multi-hypothesis tracking, finite set statistics, and outer probability measure theory, to present the various forms of risk, all based on a single general and rigorous definition that enables the handling of many of the above unique situational and data aspects of individual collision scenarios. The intent of this paper is not to answer questions, per se, but to expose readers to the technical aspects and the operational benefits and feasibility of using risk as an analytical framework for tackling many collision assessment and avoidance scenarios in Space Traffic Management.

1. INTRODUCTION

Each space conjunction event is unique. This uniqueness arises not just from the situational specifics pertaining to the conjunction at hand, such as the number and maneuverability of the objects in question, but also the nature of the available relevant data. The quality, completeness and our degree of trust in the data all contribute to the uniqueness of each event. Furthermore, current practice employs computations that assume data that is only corrupted by sensor noise, ignoring the fact that the data may be incomplete or we lack trust in it altogether. Finally, decisions based on such an analysis rely on employing a single, pre-established probability of collision threshold that ignores the unique realities of each individual conjunction.

Ignoring the unique aspects of a collision event, as well as the utilization of a "one size fits all" probability of collision threshold, can result in suboptimal conjunction assessments that are either too conservative (too many false positives) or too permissive (too many false negatives). A false positive, in this context, is when a "collision is likely" decision is made when one is less likely to happen, and a false negative is when a "collision not likely" decision is made when one is more likely to happen. The over-conservative nature of a probability of collision based threshold has typically been overlooked in the past due to the relatively low space object population density, and due to the overall trust in and the week assumptions made about the data. Moreover, the rapidly increasing number of satellites in orbit and the diversity of available data are resulting in a more complex decision-making process and trade space. We will no longer have the luxury of subjecting individual conjunction events to an arbitrarily chosen threshold, nor of assuming that data will be complete or trustworthy.

To address the unique data and situational aspects of each conjunction, the broader Space Traffic Management (STM) community should embrace mathematically formal and subjective risk/benefit based analysis for conjunction assessment and decision-making. The risk (equivalently, benefit) associated with a decision is defined as the total sum of the cost (equivalently, reward) of an event multiplied by its probability of occurrence, summed over all possible events. For example, the risk associated with a "do not maneuver" decision is the sum of (a) the cost of not maneuvering and a collision occurring multiplied by the probability that a collision occurs, and (b) the cost of not maneuvering and a collision not occurring multiplied by the probability that a collision does not occur. The resulting value reflects the risk associated with a "do not maneuver" decision. A similar computation is performed to assess the risk associated with a "maneuver" decision. The decision that yields a lower risk is the final decision to be made. The analysis is also prescriptive as it answers not just whether to maneuver, but also how to maneuver if a maneuver is deemed necessary. Replacing a threshold-based decision criterion with a risk-optimal criterion tailors the decision-making process to the very unique situational aspects of the collision event at hand.

The above description of the risk/benefit analysis is a simplified one and that we will expand on in this paper. The calculation can grow in sophistication and complexity, yet remain at mission-level time-lines, especially in regards to the nature and handling of data at hand. A risk-based analysis can accommodate: uncertain but known (multi-)object locations; uncertainty associated with the number and locations of objects; incomplete or altogether missing data (as in the case of known lost objects); and trust in the data. That is to say, a risk-based analysis is amenable to an analysis where we have both aleatoric and epistemic forms of uncertainty. Aleatoric uncertainty describes truly random variability present in physical processes, while epistemic uncertainty reflects the analyst's lack of knowledge about or trust in some aspect of the problem at hand.

All of the above underscores the uniqueness of every conjunction event and the value of employing a risk-based analysis in assessing collision scenarios. This is especially relevant in light of the introduction of mega-constellations, and in the diversity of the quality, completeness and trust aspects of the available data. In place of current practice, in this paper we propose a mathematically formal and subjective risk/benefit analysis for conjunction assessment and decision-making. Such an analysis replaces the use of a pre-determined probability of collision threshold with a rational criterion, optimized around the unique aspects of the situation at hand, for how best to act in the face of a potential collision. A risk-based analysis can accommodate highly complex decision environments and can accommodate the unique aspects of the relevant data including but not limited to: uncertain but known (multi-)object locations; uncertainty associated with the number and locations of objects; incomplete or altogether missing data (as in the case of known lost objects); and trust in the data. Specifically, in this paper we draw from various mathematical frameworks, such as multi-target tracking, multi-hypothesis tracking and finite set statistics, and outer probability measure theory, to present the various forms of risk, all based on a single abstract definition, that enable the handling of many of the above unique situational and data aspects of individual collision scenarios. The intent of this paper is not to answer questions, per se, but to expose readers to the technical and operational benefits and feasibility of using risk as an analytical framework for tackling many collision assessment and avoidance questions in STM.

There has been several proposals into how to handle collision scenarios that do not depend on a probability of collision threshold criterion. For example, Ref. [1] use Wald's Sequential Probability Ratio Test [2] to inform decisions concerning collision risk mitigation maneuvers and attempt to optimize between false positive and false negative rates. The authors, however, only ask the question of whether there is sufficient data to act on a collision situation. The methodology by the authors does not address the consequences of the maneuver action –i.e., it does not consider risk as a metric in the decision making process. The methodology does not prescribe how to maneuver, either, if one is deemed necessary. An interesting follow up question to the ideas presented in this paper is how to integrate the risk metrics discussed here with Wald's SPRT to produce a more nuanced collision avoidance decision-making analysis that first asks whether we have enough data to decide, as per the Wald test, and if we do, whether we should maneuver or not and how, taking into account the consequences of the maneuver decision. The authors in Ref. [3] have combined Wald's SPRT with a risk-based framework for autonomous underwater mobile sensor tasking. This paper seeks to develop a similar methodology but for the space collision avoidance problem, where the decisions are about whether and how a spacecraft should be maneuvered (autonomously or not) to mitigate the costs associated with a collision scenario. We note here that while some work, such as that in Ref. [4], uses the term risk in their analysis, the analysis itself is devoid on any explicit considerations of the consequences of collision decisions and their costs. A similar comment applies to the report in Ref. [5] that discusses risk in the context of the collision mitigation problem.

The paper is organized as follows. In Section (2), we appeal to an intuitive example from finance to establish the most basic definition of risk. In Section (3), we use this basic notion of risk to introduce a naïve definition of risk applied to space conjunction assessment and decision-making. This naïve definition of risk uses the conventional mathematical definition of probability of collision and can only answer the question of *whether* to maneuver, but not *how* to maneuver. A more general definition of risk, however, not only does not require the explicit computation of the probability of collision, but also provides a prescription of how to optimally maneuver if one is deemed necessary. This more general, formal risk definition is introduced in Section (4). In Section (5), we further extend the formal definition to handle complex environments, that include scenarios such as collision assessment in the presence of multiple targets, both of known and unknown numbers. Finally, in Section (6), using outer probability measure theory, we extend the analysis to include the cases when relevant data is missing, and not just uncertain. This is particularly relevant to situations such as when there is one or more lost objects. We conclude the paper with a summary of the work presented in this paper and direction for further research in Section (7).

2. QUANTITATIVE RISK ANALYSIS

We start our discussion with an appeal to quantitative risk analysis in finance. This is a natural starting point due to its intuitiveness and every-day familiarity to most readers as we build up towards a STM-relevant risk definition in the next section. There is no single definition of risk. Economists, behavioral scientists, risk theorists, statisticians, actuaries, and historians each has her/his own concept of risk [6]. In this section we will adopt a very simplistic and basic definition that aligns closely with an intuitive definition useful for the STM problem. In finance, inputs to a risk analysis are (a) a model that represents how the situation in question may evolve under different scenarios, and (b) the decision variables that the analyst needs to optimize given the model in (a). The analysis can be deterministic or probabilistic in nature. We will focus on the latter given the fact that uncertainty and ignorance[1] play a crucial role in most real-world problems, including in STM. While the term “risk” is used to connote an undesirable (negative) outcome, a desirable (positive) outcome is given the term “benefit”'. For the sake of convenience, we will use “risk” to connote a full risk/benefit analysis.

Mathematically, the risk associated with a decision is defined as the total sum of the cost of an event multiplied by its probability of occurrence, summed over all possible events. At a fundamental level, an event here is a pair composed of a decision made by the analyst and an associated outcome that is based on that decision. Let’s consider a simple financial example. Assume that an analyst is to decide whether to invest in a stock that is currently valued at \$20. There are then two decisions: (a) Decision A_1: invest (i.e., spend \$20 on the stock), and (b) Decision A_2: Do not invest (do not spend \$20 on the stock). Assume further that there are only two scenarios: (i) Outcome X_1: stock value increases to \$40, and (ii) Outcome X_2: stock value drops to \$10. Let us further assume that the probability of the stock value increasing is 20% and to drop is 80%. The risk associated with Decision A_1 is then given by:

$$\begin{aligned} R(A_1) &= p(X_1)\cdot C(X_1|A_1) + p(X_2)\cdot C(X_2|A_1) \\ &= 0.2\times(20-40) + 0.8\times(20-10) = +4, \end{aligned}$$

where $p(X_i)$ is the probability of scenario X_i materializing, and $C(X_i|A_j)$ is the cost associated with scenario X_i given Decision A_j, $i, j = 1,2$, was made. Similarly, the risk associated with Decision A_2 is then given by:

$$\begin{aligned} R(A_2) &= p(X_1)\cdot C(X_1|A_2) + p(X_2)\cdot C(X_2|A_2) \\ &= 0.2\times(40-20) + 0.8\times(10-20) = -4. \end{aligned}$$

The reader may note that the risk associated with an action A_j, $j = 1,2$, is the expected value of cost over all possible scenarios $X \in \{X_1, X_2\}$, $j = 1,2$:

$$R(A_j) = E_{p(X)}\left[C(X|A_j)\right],\ j = 1,2. \tag{1}$$

This is an observation that we will revisit later in the paper. In this example, the optimum risk-based decision would then be to not invest in the stock since the risk associated with that decision is lower than that associated with investing. This is primarily because the probability of it losing value is significantly higher than its value increasing. Note that if the stock value were to increase to, say, \$100 the decision would be reversed because the higher likelihood of it losing

[1] The delineation between uncertainty and ignorance will be made later in the paper.

value is trumped by the significantly higher potential reward ($100 instead of $40), making the investment worthwhile. Similarly, let us say that the cost of the stock is $0 (e.g., it was given as a gift), then the decision not to invest would also be reversed because the worst that can happen is for one to get back $10 at the time of sale –the receiver of the stock willl come out with a net positive return even in the worst case scenario drops in value.

Realistic quantitative risk analyses in finance are, of course, far more complex than in the above example. At minimum, for example, an action, once taken, can alter event probabilities. Situation behavior is also usually highly complex and nonlinear. Monte Carlo (MC) simulations are usually employed to generate a range of possible outcomes of a decision made or action taken. The final outcome from the MC simulations is a probability distribution of all possible outcomes that are then summarized using measures of central tendency such as the mean, median and variance. This provides a picture to the decision maker who then proceeds to make a decision based on her degree of risk tolerance or aversion.

3. NAÏVE RISK-BASED COLLISION ASSESSMENT

Given the basic understanding of risk presented in the previous section, we now apply the same basic notion of risk to the collision assessment problem. In the rest of this paper, we assume that there is only one spacecraft that we own and can control, called "the spacecraft", while other nearby space objects that we have no control over will be called "resident space objects" (RSOs). The analysis presented in this paper can be extended to the situation where the analyst is assessing the risks and courses of actions for two or more objects over which she has control. It can also be extended to the case where RSOs in the vicinity are potentially also maneuverable and are performing a similar but separate, cooperative or non-cooperative, analysis. This introduces a game theoretic aspect to the analysis that we do not address in this paper.

Present day collision assessment practice, approaches the problem by first computing the object-on-object collision probability, P_c[2]. If the computed P_c is lower than a given, pre-defined threshold no action is taken. If, on the other hand, P_c is higher than the threshold, owners/operators of one or both objects are notified who, in turn, decide on whether to maneuver. Present day collision assessment practice, approaches the problem by first computing the object-on-object collision probability, P_c[3]. If the computed P_c is lower than a given, pre-defined threshold no action is taken. If, on the other hand, P_c is higher than the threshold, owners/operators of one or both objects are notified who, in turn, decide on whether to maneuver.

This practice, however, can result in suboptimal conjunction assessments that are either too conservative (too many false positives) or too permissive (too many false negatives), as discussed in Section (1). In this section, we present a naïve risk-based analysis framework that does not rely on a simplistic threshold-based decision criterion and that, instead, optimizes the decision to act based on a rational cost/benefit analysis similar to the nalysis in the previous section. We call this analysis naïve because it focuses on only the two objects that are potentially on a collision course, as traditionally done in present day practice, ignoring any other neighboring RSOs that may influence the assessment outcomes. It is also assumed that P_c has already been computed before the analysis and is provided from an external source. Finally, this naïve analysis can at best address the question of whether to maneuver, but not how to maneuver. In the next section, we will step back and provide a general mathematical definition for risk that implicitly computes a weighted version of P_c (weighted by, effectively, the distribution of the cost of a collision occurring at all probable collision locations), and that will enable the accommodation of far more complex situational and data quality scenarios than the ones considered in this section, including providing a prescription on how to maneuver if maneuvering is deemed the risk-optimal choice.

Let us assume that we are given P_c. Instead of a threshold-based criterion for decision-making, in a risk-based analysis we ask the question: what are the costs associated with (a) unnecessarily maneuvering to avoid a collision that may not actually occur (costs of a false positive), and (b) erroneously not maneuvering in an attempt to save fuel when a collision that subsequently does actually occur (costs of a false negative). Using the same notation from the last section, the risks associated with the two decisions (to maneuver or not to maneuver) can then be computed as follows:

$$R(\text{Do Not Maneuver}) \quad = \quad (1-P_c)C(\text{No Collision}|\text{Do Not Maneuver}) + P_c C(\text{Collision}|\text{Do Not Maneuver})$$

[2]While there is much that can be done by the community to improve the fidelity of computation of P_c, this is a topic that is beyond the scope of this paper.

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$$R(\text{Maneuver}) \quad = \quad (1-P_c)C(\text{No Collision}|\text{Maneuver}) + P_cC(\text{Collision}|\text{Maneuver}), \tag{2}$$

where $1-P_c$ is the probability that a collision does not occur. $C(\text{No Collision}|\text{Do Not Maneuver})$ is the cost associated with not maneuvering and no collision occurring. This cost is practically zero, unless one wants to account for the psychological anxiety of not maneuvering when the probability of collision is nonzero. $C(\text{Collision}|\text{Do Not Maneuver})$ is the cost associated with a collision occurring given that a maneuver was not executed. This cost is composed of the loss of the satellite, associated loss of service, any potential legal implications of a collision as a consequence of not maneuvering, and so on. $C(\text{No Collision}|\text{Maneuver})$ is the cost associated with maneuvering and a collision not occurring. This cost is subjective and can depend on factors such as how much fuel expenditure is required by the maneuver compared to the total available on-board the satellite (i.e., fuel availability on the margin). Finally, $C(\text{Collision}|\text{Maneuver})$ is the cost associated with maneuvering and still colliding with the other object.

Eq. (2) can now be optimized with the optimization variables being whether to maneuver or not. The decision that minimizes risk will then be chosen. What the above definition of risk lacks is the ability to prescribe how to maneuver, once a maneuver decision is deemed optimal. It also ignores the fact that cost can depend on the way we choose to maneuver. Both of these deficiencies will be addressed in the next section via a more formal and a more general definition of risk.

4. FORMAL RISK-BASED COLLISION ASSESSMENT

In the previous section, two assumptions were made. The first assumption was that the probability of collision P_c was already computed elsewhere and then provided to the analyst to perform her analysis. The second assumption is that the decision space is primarily a discrete one: to maneuver or not to maneuver. In this section we provide a formal risk definition that addresses both assumptions. First, the proposed risk definition implicitly computes the probability of collision and, in doing so, results in a higher fidelity assessment of risk. Secondly, unlike the analyses in the previous two sections, where decisions are of a discrete nature, if we think of the collision resolution problem as one of deciding not just *whether* to maneuver, but also *how* to maneuver, the decision variable is now continuous. The formal definition of risk, thus, provides the mechanics for assessing the optimal way to maneuver if a maneuver is deemed necessary. We note here that the question of *how* to maneuver also answers the question of whether to maneuver at all –an optimal maneuver that entails a "zero" maneuver is a decision not to maneuver at all.

We introduce the mathematical definition of risk by way of appealing to the original definition of the probability of collision. The probability of collision, P_c, is mathematically defined as the integral of the probability density representing the chances that two objects occupy the same volume V [7]:

$$P_c = \int_V p_X(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{3}$$

where $\mathbf{x}$ is the relative position vector between the centroids of the two objects, $p_X(\mathbf{x})$ is the probability density function for $\mathbf{x} \in X$, and V is the volume over which we desire to compute the probability of collision, which is typically a spherical volume whose diameter is the sum of the hard body radii, d, of the two objects. The volume V is typically defined at the anticipated point of closest approach.

Going from the above definition of P_c to risk is relatively straightforward. But first, let's address the issue of continuity versus discreteness as per the two examples given in Sections (2) and (3). In general, for the discrete case, if we have N discrete actions to choose from, say A_i, $i = 1,\ldots,N$, then there are N risks to be computed, one per decision, denoted by $R(A_i)$, $i = 1,\ldots,N$. Extending this notion to the continuous case, we would then obtain a risk function $R(\mathbf{u})$, $\mathbf{u} \in \mathscr{U}$, where we have switched the notation from the set of discrete actions $\mathscr{A} = \{A_i, i = 1,\ldots,N\}$ to the space $\mathscr{U}$ of candidate thrust vectors, represented by the variable $\mathbf{u}$. The space $\mathscr{U}$ can represent complex models for the set of controls. For example, $\mathscr{U}$ may represent the set of continuous controls over a window of time, or it may represent the class of a series of finite impulses over a window of time. For the purposes of this paper, we need not assume any one particular class of controls.

Returning to the definition of risk, with a probability density, $p_X(\mathbf{x})$, representing the random uncertainty in the relative position vector between the spacecraft and another RSO, we can replace the discrete cost functions $C(\mathrm{O}_i|\mathrm{A}_j)$ with a continuous cost function $C(\mathbf{x}|\mathbf{u})$. Effectively, $C(\mathbf{x}|\mathbf{u})$ describes what the cost is for every relative location vector $\mathbf{x}$ given action $\mathbf{u}$. Instead of a discrete sum, we would then have an integral of the product of the probability density and the cost function, giving us the formal definition of risk as a function of the maneuver decision $\mathbf{u}$:

$$R(\mathbf{u}) = \int p_X(\mathbf{x}) C(\mathbf{x}|\mathbf{u}) \mathrm{d}\mathbf{x}. \tag{4}$$

We note here that the integral is over the entire space of possible values of $\mathbf{x}$. Whereas in the classical calculation of the probability of collision we limit ourselves to a volumetric integral over the combined hard body, Eq. (4) seeks to compute the consequence of a collision occurring *anywhere* in space.

The cost $C(\mathbf{x}, \mathbf{u})$ could, for example, be of the form:

$$C(\mathbf{x}, \mathbf{u}) = c_u(\mathbf{u}) + c_s \cdot \mathbf{1}_d(\mathbf{x}), \tag{5}$$

where $c_u(\mathbf{u})$ is a function modeling the amount of fuel spent associated with $\mathbf{u}$, c_s is the cost, for example, associated with the loss of the spacecraft, services, etc., and $\mathbf{1}_d(\mathbf{x})$ is the spherical indicator function centered at $\mathbf{x} = \mathbf{0}$ and with diameter equal to the combined hard body diameter d. Finally, given the distribution $p_X(\mathbf{x})$ and the cost $C(\mathbf{x}|\mathbf{u})$, the optimum decision $\mathbf{u}^*$ that minimizes risk is then given by

$$\mathbf{u}^* = \mathrm{argmin}_{\mathbf{u}} R(\mathbf{u}) \tag{6}$$

Returning to the integral in Eq. (4). This expression shares another quality with the discrete definition of risk that was given in Eq. (1): Risk associated with a decision is the expected value, in the probabilistic sense, of the cost associated with the decision. The expectation is taken over the distribution describing the statistics of the underlying random variables. In the naïve case, this is the event space composed of $\{\text{Collision, No Collision}\}$. In the continuous case, this is the random variable $\mathbf{x}$. We quickly see, then, that the risk $R(\mathbf{u})$ can equivalently be expressed as:

$$R(\mathbf{u}) = E_{p_X(\mathbf{x})}\left[C(\mathbf{x}|\mathbf{u})\right], \tag{7}$$

where $E_{p_X(\mathbf{x})}\left[f(\mathbf{x})\right]$ is the expected value of $f(\mathbf{x})$ with respect to the distribution $p_X(\mathbf{x})$. This is an important observation since it provides us with a prescription on how to numerically solve the optimization problem in Eq. (6). Appealing to the law of the unconscious statistician, a Monet Carlo based estimate of the integration in Eq. (6) is then given by:

$$R(\mathbf{u}) = \int p(\mathbf{x}) C(\mathbf{x}|\mathbf{u}) \mathrm{d}\mathbf{x} \simeq \frac{1}{\mathrm{L}} \sum_{\mathrm{i}=1}^{\mathrm{L}} \mathrm{C}(\mathbf{x}_{\mathrm{i}}, \mathbf{u}), \tag{8}$$

where $\mathbf{x}_i$, $i = 1, \ldots, L$, are L samples drawn from the probability density function $p(\mathbf{x})$. Monte Carlo integration is very attractive given that it is a highly parallelizable computation.

5. RISK-BASED DECISION MAKING IN COMPLEX ENVIRONMENTS

In the previous section, we looked at the case where the satellite is potentially colliding with only one other RSO. In this section, we look into the situation where the spacecraft is potentially on a collision course with several other RSOs. The situation considered here also pertains to the case where the spacecraft is potentially on a collision course with only one RSO –the main RSO– but there are other RSOs in the vicinity with which it can collide as a consequence of the maneuver it executes to avoid a collision with the main RSO. In either of these cases, on the data side, two other scenarios need to be considered in the analysis. The first scenario is when there are multiple RSOs to account for in the analysis, with a known number of RSOs. The second scenario is when there are multiple RSOs to account for in the analysis, but the number of RSOs is uncertain. The former scenario entails a straightforward extension of the analysis from the last section to the multi-target case via a multi-target joint probability density function. The latter scenario requires machinery such as Random Finite Sets (RFSs) and Finite Set Statistics (FISST) [8]. Starting with the observation that risk is the expected value of cost, in the next two subsections we will extend the analysis from the previous section to handle the multi-target situation with known and unknown number of objects.

5.1 Risk Analysis with a Known Number of Objects

Extending from the single object to the multiple object case is as simple as replacing the single probability density $p_X(\mathbf{x})$ with the joint density $p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$. The risk function in this case would then be:

$$R(\mathbf{u}) = \int p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) C(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N | \mathbf{u}) \mathrm{d}\mathbf{x}. \tag{9}$$

The cost function $C(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N|\mathbf{u})$ reflects the costs associated, essentially, with different collision scenarios –e.g., with only one of the N RSOs, or with a subset of them that are closely spaced, etc. We see that the cost model C can rapidly grow in complexity, but once a model is arrived at, the analysis proceeds exactly as before. This is especially so if it is easy to sample the density $p(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$. While the problem can grow in scale, we have a highly parellelizable integration technique, Monte Carlo integration, to evaluate risk.

Next, let us consider a scenario that is expected to become dominant as mega-constellations get deployed. First, it is safe to assume that the uncertainty relative locations of the RSOs are independent. Eq. (9) can then further be written as:

$$R(\mathbf{u}) = \int p(\mathbf{x}_1) p(\mathbf{x}_2) \cdots p(\mathbf{x}_N) C(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N | \mathbf{u}) \mathrm{d}\mathbf{x}_1 \cdots \mathrm{d}\mathbf{x}_\mathrm{N}. \tag{10}$$

The independence assumption will not hold, for example, when objects are too closely spaced for the perceiving sensors to be able to discern which observation came from which RSO. In other words, the assumption will not hold when we are unable to resolutely associate sensors observations to unique objects. In that case, one can pursue techniques such as FISST that will be discussed in the next section.

Next assume that that the RSOs in the vicinity of the spacecraft are well-separated, but still sufficiently close to make the risk analysis meaningful. In this case, an imminent potential collision can be with one or several of them. Sufficient separability result in the following property:

$$C(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N|\mathbf{u}) \quad = \quad c_u(\mathbf{u}) + C(\mathbf{x}_1|\mathbf{u}) + C(\mathbf{x}_2|\mathbf{u}) + \cdots + C(\mathbf{x}_N|\mathbf{u}),$$

where $c_u(\mathbf{u})$ is as defined in the previous section and where each of the terms $C(\mathbf{x}_i|\mathbf{u})$, $i = 1, \ldots, N$, represents the cost associated with the loss of service and spacecraft in case of a collision with RSO i. For example, one can assume that $C(\mathbf{x}_i|\mathbf{u}) = c_s \cdot \mathbf{1}_d(\mathbf{x}_i)$, $i = 1, \ldots, N$, as in the previous section. This property comes from the fact that the RSOs are well separated, and, thus, that the costs associated with the different potential collisions between the spacecraft and the RSOs are non-overlapping, for any given control $\mathbf{u}$. With this observation Eq. (10) then simplifies to become:

$$\begin{aligned}
R(\mathbf{u}) &= c_u(\mathbf{u}) + \int p(\mathbf{x}_1) p(\mathbf{x}_2) \cdots p(\mathbf{x}_N) C(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N|\mathbf{u}) \mathrm{d}\mathbf{x}_1 \cdots \mathrm{d}\mathbf{x}_\mathrm{N} \\
&= c_u(\mathbf{u}) + \int p(\mathbf{x}_1) p(\mathbf{x}_2) \cdots p(\mathbf{x}_N) \left[C(\mathbf{x}_1|\mathbf{u}) + C(\mathbf{x}_2|\mathbf{u}) + \cdots + C(\mathbf{x}_N|\mathbf{u}) \right] \mathrm{d}\mathbf{x}_1 \cdots \mathrm{d}\mathbf{x}_\mathrm{N} \\
&= c_u(\mathbf{u}) + \sum_{i=1}^{N} \int p(\mathbf{x}_1) p(\mathbf{x}_2) \cdots p(\mathbf{x}_N) C(\mathbf{x}_i|\mathbf{u}) \mathrm{d}\mathbf{x}_1 \cdots \mathrm{d}\mathbf{x}_\mathrm{N} \\
&= c_u(\mathbf{u}) + \sum_{i=1}^{N} \left\{ \int \left[p(\mathbf{x}_i) C(\mathbf{x}_i|\mathbf{u}) \mathrm{d}\mathbf{x}_\mathrm{i} \right] \cdot \left[\Pi_{j \neq i} \int p(\mathbf{x}_j) \mathrm{d}\mathbf{x}_\mathrm{j} \right] \right\} \\
&= c_u(\mathbf{u}) + \sum_{i=1}^{N} \int p(\mathbf{x}_i) C(\mathbf{x}_i|\mathbf{u}) \mathrm{d}\mathbf{x}_\mathrm{i}.
\end{aligned} \tag{11}$$

In other words, the total risk associated with a control $\mathbf{u}$ is equal to the individual risks associated with all the objects when the spacecraft is maneuvered according to $\mathbf{u}$. This outcome is, of course, intuitive: Given a control $\mathbf{u}$, the risk associated with it is simply equal to the risk associated with a collision with each of the RSOs plus the risk associated with the fuel expenditure required under the maneuver $\mathbf{u}$.

5.2 Risk Analysis with an Uncertain Number of Objects

In the above analysis, we assumed that the nearby objects, while their locations are uncertain, their number is known. The question then follows, how do we perform the analysis in case the number of objects in the vicinity of the spacecraft is unknown? This is relevant to the scenario when a potential collision with another object is imminent in the presence of a debris field in the vicinity of our spacecraft –e.g., after a recent nearby collision. The analysis presented in this section is also useful in the scenario where the objects are too closely spaced to be uniquely resolved from each other and the observations are not easily associated with the RSOs. This scenario was highlighted in the previous section as one that can not be treated using simple multivariate probability theory.

In this case, we need tools from RFS theory and Finite Set Statistics (FISST) [8]. Risk is still the expected value of a cost function C, except that in this case, the cost function is a mapping from the space of set-valued random variables $\mathscr{X}$ and the space of maneuvers $\mathscr{U}$ to the real numbers.

The risk function $R(\mathbf{u})$ is then given by:

$$R(\mathbf{u}) \quad = \quad E_{p(X)}\left[C(X|\mathbf{u})\right] = \int C(X,\mathbf{u})p(X)\delta X, \tag{12}$$

where we note that now $X \in \mathscr{X}$ and where the integral is a set integral [8]. Expanding the set integral, one gets [8]:

$$\begin{aligned} R(\mathbf{u}) \quad = \quad & p(\emptyset)C(\emptyset|\mathbf{u}) + \frac{1}{1!}\int p(\{\mathbf{x}_1\})C(\{\mathbf{x}_1\}|\mathbf{u})\mathrm{d}\mathbf{x}_1 \\ & + \frac{1}{2!}\int p(\{\mathbf{x}_1,\mathbf{x}_2\})C(\{\mathbf{x}_1,\mathbf{x}_2\}|\mathbf{u})\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2 + \ldots \\ & + \frac{1}{n!}\int p(\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\})C(\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\}|\mathbf{u})\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\cdots\mathrm{d}\mathbf{x}_\mathrm{n} + \ldots, \end{aligned}$$

where $p(\emptyset)$ represents the probability that there are no objects in the vicinity of the spacecraft, $C(\emptyset|\mathbf{u})$ is the cost associated with there being no objects given the control $\mathbf{u}$ (which would amount to only the cost of applying the control $\mathbf{u}$ since there is zero risk of colliding with any other objects), $p(\{\mathbf{x}_1\})$ is the probability density associated with the hypothesis that there exists only on object whose relative location with respect to the spacecraft is denoted by $\mathbf{x}_1$, $C(\{\mathbf{x}_1\}|\mathbf{u})$ is the cost associated with the control $\mathbf{u}$ and there being one RSO whose relative location with respect to the spacecraft is given by $\mathbf{x}_1$, and so on. We will not get into further details of the set theoretic analysis other than note that it eventually reduces to computations similar in nature to those described in previous sections for the non-set-theoretic case. Fundamentally, the only difference between using the set-theoretic framework and the non-set-theoretic one is that while the computations at heart are still highly parallizable, they just grow in scale due to the multi-hypothesis nature of the problem. However, techniques such as Smart Sampling Markov Chain Monte Carlo (see, for example, the work in Ref. [9]) develops approximations of the formal FISST-based formulation in such a way that make integrals such as those in Eq. (12) scalable in the number of hypotheses without resorting to moment-approximating techniques such as the Probability Hypothesis Density (PHD) [10] and the Cardinalized Probability Hypothesis Density (CPHD) [11].

It is interesting to note that if in Eq. (12) we replace $C(X,\mathbf{u})$ with the Dirac delta density $\delta(\mathbf{x})$ defined over the generic spatial variable $\mathbf{x}$, the above expression is then identical to the PHD [8]. In other words, in this case, if we integrate the PHD over $\mathbf{x}$ we end up with risk associated with no maneuver and where all collision costs are set to 1.

6. RISK ANALYSIS IN THE PRESENCE OF IGNORANCE

In this section, we consider the question of data insufficiency. Data insufficiency is when data is entirely missing, not just uncertain. An application scenario is when an object is lost. In this case, we do know that it is lost, which means that we know that it exists but do not know where it is. Philosophically, a lost object is not to be treated as a probabilistic event, as is commonly practiced. Instead, it has experienced a deterministic phenomenon either in the form a maneuver by an operator, or in the form of a space environment driven event, or a mechanical event such as an internal implosion. While these events have elements of uncertainty associated with them, the phenomena are fundamentally deterministic in nature, but are ones that we are ignorant (not uncertain) about. As such, treating our ignorance, such as in the case of a lost object location, probabilistically is not an appropriate framework. Instead, we have to resort to an alternative analysis technique that can handle ignorance. In the situation where, say, a nearby object goes missing and, concurrently, we find the spacecraft on a potential collision course with another satellite of known, but probabilistically uncertain location, the question then becomes: can we chose a control $\mathbf{u}$ that not only avoids a collision with the main RSO, but also minimizes the likelihood of colliding with the missing object whose location we do not know?

To answer this question, we need machinery that allows the handling of both uncertainty and ignorance in one single framework. In other words, we need machinery that enables us to delineate between and quantify both aleatoric and epistemic forms of uncertainty. While aleatoric uncertainty describes truly random variability present in physical processes, epistemic uncertainty describes lack of knowledge by the analyst about some aspect of the problem or system at hand. Aleatoric uncertainty is conventionally just called "uncertainty", while epistemic uncertainty is often referred to as "ignorance". The space community has been utilizing the state of the art in probability theory and uncertainty quantification to model and process aleatoric uncertainty. It has, however, barely considered the modeling and processing of epistemic uncertainty. In fact, current common practice treats epistemic uncertainty as aleatoric. This practice not only violates fundamentals of uncertainty analysis, it has important safety implications that can result in a misanalysis with potentially catastrophic operational consequences.

For example, consider how an analyst traditionally computes the probability of collision. Typically, the uncertain location of a space object is represented by a nominal time-stamped location (the mean of a Gaussian distribution) and some representation of the uncertainty associated with its location (the covariance of a Gaussian distribution, or, the "uncertainty ellipsoid"). Common practice is that if the analyst is highly uncertain about an object's location, say if it is lost, a mean location is propagated from its last known location and an arbitrarily large covariance matrix is assigned to it as an attempt to reflect the fact that the object is lost. Computing a probability of collision based on this uncertainty model will result in a very small probability of collision due to the fact that the uncertainty ellipsoid is artificially inflated. So, despite the fact that the analyst has very little idea where the lost object actually is, she arrives at the conclusion that the probability of colliding with it is negligible –for all she knows, the object could very well be close to a valuable space asset, resulting in a high potential for a catastrophic collision on orbit! While a human analyst can usually catch such paradoxes and adjust accordingly, current practice, techniques and tools do not facilitate giving a machine the ability to identify and handle such situations in an automated fashion.

There are probabilistic techniques, such as outer probability measure (OPM) theory, that enable the representation, processing and handling of ignorance that are amenable to automated implementation on a machine (see Refs. [12, 13] and references therein). In Ref. [13] for example, the authors demonstrate how to quantify and represent uncertainty, both aleatoric and epistemic, associated with the initial orbit determination (IOD) problem, in which an object's track is initialized with, possibly multi-phenomenology, sensor data that is insufficient to completely determine its track. Such an IOD analysis that properly delineates between aleatoric and epistemic uncertainty can result in much more accurate representation of our state of knowledge and that can produce outcomes vastly different from and more accurate than ones that conflate aleatoric and epistemic uncertainty.

The use of OPM theory to represent and process ignorance can be extended to many other aspects of the STM problem beyond the calculation of probability of collision or initializing the orbit of an object with incomplete data. OPM theory also lends itself naturally to risk-based decision analytics discussed in this paper. Most importantly, the theory enables a transparent, concrete and clear quantification of and delineation between what we do not know, and what we known and how (un)certain we are about it.

Let us now use the random vector $\mathbf{x}_i$, $i = 1, \ldots, N$ to denote the randomly distributed relative position vectors between the spacecraft and all nearby objects whose locations are known but are uncertain (in the random, aleatoric sense). Let $\mathbf{q}_i$, $i = 1, \ldots, M$, denote the relative position vectors between the spacecraft and objects for which we are ignorant (in the epistemic sense) about their location. The expression for risk associated with the control vector $\mathbf{u}$ given ignorance about some objects' locations and the random uncertainty about the locations of others can be expressed as

$$R(\mathbf{u}) = \int \sup_{\{\mathbf{q}_1,\ldots,\mathbf{q}_M\}} \left[p(\mathbf{x}_1,\ldots,\mathbf{x}_N) f_{\{\mathbf{q}_1,\ldots,\mathbf{q}_M\}}(\mathbf{q}_1,\ldots,\mathbf{q}_M) C(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{q}_1,\ldots,\mathbf{q}_M) \right] \mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_\mathrm{N}, \tag{13}$$

where $C(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{q}_1,\ldots,\mathbf{q}_M)$ is the cost given the locations of all $N+M$ objects, $p(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ is, as before, the joint probability density associated with the N objects with known but uncertain locations up to aleatoric randomness, and $f_{\{\mathbf{q}_1,\ldots,\mathbf{q}_M\}}(\mathbf{q}_1,\ldots,\mathbf{q}_M)$ is the *possibility function* associated with the M objects which we have complete ignorance about their locations. A possibility function $f(\mathbf{q})$ over the variable $\mathbf{q}$ is a measure of the possibility, not probability, of whether an object exists at $\mathbf{q}$. The set of possible values for $\mathbf{q}$ can be determined by the physics of orbital motion, as well as what we know about the lost object's maneuvering capabilities. The set of possible locations can, thus, be derived from a reachability analysis (see, for example, Ref. [14] and references therein). An intuitive way to understand Eq. (13) is to think of the supremum as the analogue of an integration when taking expectations in the aleatoric case. So the integration and the supremeum over the aleatoric and epistemic variables via their probability density and possibility functions, respectively, of $C(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{q}_1,\ldots,\mathbf{q}_M)$ is effectively the expectation of the cost function.

Assuming the least informative prior, i.e., where all object locations are equally possible, can be modeled using a uniformly distributed possibility function. Historically, rsearchers have typically used a uniform probability density function. This practice however is suboptimal since the former says that the object can be anywhere within the possible/reachable set with equal probability. A uniform possibility function, on the other hand, only says that the object is somewhere within the reachable set without assigning a likelihood of where in the set the object might be.

Assuming again that the objects are sufficiently well separated and that the objects have locations that are independent of each other, Eq. (13) reduces to:

$$\begin{aligned} R(\mathbf{u}) &= \int \sup_{\{\mathbf{q}_1,\ldots,\mathbf{q}_M\}} [C_1(\mathbf{r}_1)+\ldots+C(\mathbf{r}_N)+(\mathbf{q}_1)+\ldots+C(\mathbf{q}_M)] \\ &\quad \times f_{\mathbf{q}_1}(\mathbf{q}_1)\times\ldots\times f_{\mathbf{q}_M}(\mathbf{q}_M)p(\mathbf{x}_1)\cdots\times p(\mathbf{r}_N)\mathrm{d}\mathbf{r}_1\ldots\mathrm{d}\mathbf{r}_\mathrm{N}. \end{aligned} \tag{14}$$

Let us consider a simple case and see that at least the resulting expression is something one would anticipate. Consider the case when there are only two objects, one of a known but randomly distributed relative location $\mathbf{x}$ and another that is lost but with an unknown relative location $\mathbf{q}$. Eq. (14) then reduces to:

$$R(\mathbf{u}) = \int \sup_{\{\mathbf{q}\}} [C(\mathbf{r},\mathbf{q})f_{\mathbf{q}}(\mathbf{q})]\, p(\mathbf{r})\mathrm{d}\mathbf{r}, \tag{15}$$

where subscripts are eliminated for simplicity, with $\mathbf{q}$ being the location of the lost object and $\mathbf{r}$ being the location of the other object. If, as above, the two objects are sufficiently separated –i.e., that the known object is guaranteed to be outside the reachable/possible of the lost object location, then Eq. (15) reduces to:

$$\begin{aligned} R(\mathbf{u}) &= \int \sup_{\{\mathbf{q}\}} [(C(\mathbf{r})+C(\mathbf{q}))\, f_{\mathbf{q}}(\mathbf{q})]\, p(\mathbf{r})\mathrm{d}\mathbf{r} \\ &= \int \sup_{\{\mathbf{q}\}} [C(\mathbf{r})f_{\mathbf{q}}(\mathbf{q})]\, p(\mathbf{r})\mathrm{d}\mathbf{r} + \int \sup_{\{\mathbf{q}\}} [\mathrm{C}(\mathbf{q})\mathrm{f}_{\mathbf{q}}(\mathbf{q})]\, \mathrm{p}(\mathbf{r})\mathrm{d}\mathbf{r} \\ &= \int \sup_{\{\mathbf{q}\}} [f_{\mathbf{q}}(\mathbf{q})]\, C(\mathbf{r})p(\mathbf{r})\mathrm{d}\mathbf{r} + \int \sup_{\{\mathbf{q}\}} [\mathrm{C}(\mathbf{q})\mathrm{f}_{\mathbf{q}}(\mathbf{q})]\, \mathrm{p}(\mathbf{r})\mathrm{d}\mathbf{r} \\ &= \sup_{\{\mathbf{q}\}} [f_{\mathbf{q}}(\mathbf{q})] \int C(\mathbf{r})p(\mathbf{r})\mathrm{d}\mathbf{r} + \sup_{\{\mathbf{q}\}} [\mathrm{C}(\mathbf{q})\mathrm{f}_{\mathbf{q}}(\mathbf{q})] \int \mathrm{p}(\mathbf{r})\mathrm{d}\mathbf{r} \\ &= \int C(\mathbf{r})p(\mathbf{r})\mathrm{d}\mathbf{r} + \sup_{\{\mathbf{q}\}} [\mathrm{C}(\mathbf{q})\mathrm{f}_{\mathbf{q}}(\mathbf{q})] \\ &= R_{\text{uncertain}}(\mathbf{u}) + R_{\text{lost}}(\mathbf{u}), \end{aligned} \tag{16}$$

where $R_{\text{lost}}(\mathbf{u})$ is the risk associated with a collision with the lost object and $R_{\text{random}}(\mathbf{u})$ is the risk associated with a collision with the object with a known but random location, both under the control $\mathbf{u}$. The outcome in Eq. (16) is exactly what one would expect: the total risk associated with a maneuver decision $\mathbf{u}$ is the sum of the risk of colliding with the lost object, and the risk of colliding with the object of known, but uncertain, location.

Finally, as discussed in, say, Ref. [13], a possibility function can be sampled in ways similar to and dependent on associated probability density functions. As such, the Monte Carlo approach discussed in Section (4) holds in this case as well. The discussion of risk in the presence of ignorance presented in this section is not intended to be thorough or exhaustive. Instead, the goal was to demonstrate that there is machinery that can enable a risk analysis even in the presence of ignorance (things we know that we do not know). A more comprehensive presentation and analysis will be presented in a future research publication.

7. CONCLUSION

In this paper we introduced advanced mathematical formulations for a risk/benefit analysis for conjunction assessment and decision-making. Such an analysis replaces the use of a pre-determined probability of collision threshold with a rational criterion that is optimized around the unique aspects of the situation at hand, and how best to act in the face of a potential collision. A risk-based analysis can accommodate highly complex decision environments and can accommodate the unique aspects of the relevant data including but not limited to: uncertain but known (multi-)object locations; uncertainty associated with the number and locations of objects; and incomplete or altogether missing data (as in the case of known lost objects). Specifically, in this paper we drew from various mathematical frameworks, such as multi-target tracking, multi-hypothesis tracking and finite set statistics, and outer probability measure theory, to present the various forms of risk, all based on a single abstract definition, that enable the handling of many of the above unique situational and data aspects of individual collision scenarios. We further described how to solve for risk, in all of the cases presented, using Monte Carlo integrations that render risk based analytics highly parallelizable and implementable at mission-level time scales.

The ideas presented in this paper were general and theoretical in nature. Further research will focus on using high-fidelity simulations to demonstrate the efficacy and advantages of using risk-based analytics over the present day practice of using the probability of collision with a pre-set threshold for collision assessment. Further analysis can be performed to refine the analysis presented in this paper to specialized cases of peak operational value. Future research will also focus on relaxing some of the assumptions made in this paper. These include expanding the analysis to specific classes of models for the space of controls $\mathscr{U}$ (e.g., the class of a series of impulsive maneuvers or the class of low-thrust continuous maneuvers).

Furthermore, the analysis presented in this paper assumed an instantaneous risk calculation that is based the anticipated uncertainty state at the time of closest approach to each individual object. A more rigorous evaluation would, instead, evaluate the cumulative risk associated with the control $\mathbf{u}$ throughout the entire maneuver window, starting from the current instant until a reasonably selected terminal time. This will naturally require integration into and with a sequential estimation framework, especially in the context of sequential Bayesian estimation theory. The Bayesian framework is particularly interesting as well, as we start tackling the problem of complexity, where several satellite owners/operators are performing similar risk-based analytics. Risk analysis and decision making in such a context may call for risk analysis and decision making in complex Bayesian networks [15].

Other extensions include solving for the optimal maneuver of not just one spacecraft, but multiple maneuverable spacecraft. Another interesting and important is when other spacecraft can maneuver and either communicate that maneuver with us to account for in our risk analysis, or does not communicate that information to us. The former scenario results in cooperative game theoretic risk-based analysis, while the latter results in a non-cooperative or adversarial games. Finally, and perhaps most importantly, much work still needs to be done to demonstrate that, from a policy and an operational perspective, using risk is far more advantageous than present day practices.

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