A Novel Analytical Method to Determine Future Close Approaches between Satellites

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ABSTRACT

Satellite operators are overwhelmed by proximity warnings published in conjunction analysis reports. Historically proven, the vast majority are false warnings and can be safely ignored. This is an artifact of using TLEs or other data sources which come with poor covariance. In turn, large thresholds are used to compensate for this uncertainty. Only by reducing the threshold for warnings paired with using more accurate data sources can we provide satellite operators an actionable list of vulnerabilities. This paper builds upon the fundamental concepts laid out in [4] and defines a new mutual frame from which equations for the relative dynamics of satellites can be described. These equations build into a new time prefilter, leading to computational savings. When comparing 6200 orbits to each other, utilizing both time prefilters was able to reduce the time windows needed to be investigated using by propagation by around half compared to just the originally proposed time prefilter.

1. EXISTING THEORY

There are two primary approaches towards conjunction analysis, series and parallel; with the fundamental papers being [4] and [8], respectively. Hoots et all in [4] suggests the limited case of comparing one satellite versus another. Here, a perigee-apogee test and a second geometric prefilter are suggested. The first determines if the two satellites have any overlapping altitude ranges, while the second is effectively a Minimum Orbit Intersectional Distance (MOID) algorithm. This is the process of taking the first 5 orbital elements and calculating the minimum distance between two orbits, irrespective of time or where the satellite is on either orbit. Lastly, a time prefilter is presented putting limits on the true anomalies of the satellite by considering the normal distance of satellites with respect to the partner orbit. In the case of non-coplanar orbits this time prefilter is very effective, however, in the case of satellites in constellations who occupy the same orbital planes, this has no positive effect. Notably, Hoots defines the problem using spherical geometry, whereas, in this paper, I will propose a mutual reference frame with new anomalies that are related to the line of nodes of the two orbits in question.

There have been many improvements and further research to each of the elements of Hoots's paper. In [10], the issue of data source was discussed and the impact of using mean versus osculting elements, specifically with respect to the apogee-perigee filter. These findings were further discussed in [2], which suggested improvements to both of the geometric filters, specifically highlighting the work by Grochi in [6] [5] [7].

The parralel approach by Healy in [8] takes advantage of the nature of parallel processes in computers to save significant cost on calculation time. This approach requires prepropagating satellites at a given interval and then using vectorized functions to calculate the difference between the prepropogated ephemeris. In the case where ephemeris already exists or using a method that is very cheap to propagate, such as TLEs, this method allows for significant cost savings. However, as the data source becomes more accurate and slower to propagate, this method fails to save time and could even become more costly since the mesh of times would include many points and fails to benefit from the prefilters outlined in [4]. Additionally, using this method, there is a chance that any pass between satellites under the given distance threshold that takes less time than the time difference in the time mesh is not noticed. Theoretically, objects with a high relative inclination could have a high-aspect, high-velocity pass that gets overlooked.

Other suggested improvements include using a smart sieve approach [9], splining [1], or chebyshev polynomials to collapse upon times of closest approach (TCAs) [3]. As these methods rely upon the windows after the prefilters, they will not be discussed as they exist further in conjunction assessment algorithm with respect to what is being proposed.

2. NEW THEORY

2.1 New Frame

As Gronchi pointed out in [5], any two orbits can be simplified to a set of 9 relative orbital elements. In this section, those 9 orbital elements will be explicitly defined via a new mutual reference frame based upon the following fundamentals.

1. We chose one orbit to be the primary, denoted by a, while the other is the secondary, denoted by b. When comparing any two orbits, there is no convention. Any orbit can be the primary or secondary.
2. We build our frame such that the z-axis is aligned with the specific angular momentum ($\vec{h}_a$) of the primary orbit
3. We align the x-axis with the line of intersection of the two orbits ($\vec{\chi}$).
4. Lastly, the y-axis closes out the right-handed frame.

The line of intersection between two orbits can be determined by taking the cross product of the two specific angular momentum vectors.

$$\vec{\chi} = \vec{h}_a \times \vec{h}_b \tag{1}$$

Utilizing the fact that ($\vec{\chi}$) is along the intersection line of the two orbital planes, we can use it to define a nodal anomaly (ψ). For any given time, the nodal anomaly can be calculated by the following equation.

$$\psi_{a,b} = \arccos \frac{\vec{r}_{a,b} \cdot \vec{\chi}}{|\vec{r}_{a,b}||\vec{\chi}|} * sign(\vec{h}_{a,b} \cdot (\vec{\chi} \times \vec{r}_{a,b})) \tag{2}$$

Next, we define the relative inclination (δ) between two orbits. This value is equal to the angle between the two specific angular momentum vectors, representative of how non-coplanar they are.

$$\cos\delta = \frac{\vec{h}_a \cdot \vec{h}_b}{h_a h_b} \tag{3}$$

We can also describe the relative inclination in terms of the traditional inclinations and RAANs.

$$\cos\delta = \sin i_a \sin i_b \cos(\Omega_a - \Omega_b) + \cos i_a \cos i_b$$

Lastly, $\tilde{\omega}$ is the angle from $\vec{\chi}$ and $\vec{e}$ in a similar manner to equation 2, allowing us to establish the following relation.

$$\psi = \tilde{\omega} + \theta$$

Combining all of these values together, we are able to map the original set of orbital elements to a reduced set.

$$a_a, e_a, i_a, \Omega_a, \omega_a, \theta_a \Rightarrow a_a, e_a, 0, \tilde{\omega}_a, \psi_a$$
$$a_b, e_b, i_b, \Omega_b, \omega_b, \theta_b \Rightarrow a_b, e_b, \delta, \tilde{\omega}_b, \psi_b$$

2.2 New Equations of Motion

Utilizing the new reduced element sets, we can define the position of the two objects as

$$\begin{aligned} \vec{r}_a &= r_a[\cos\psi_a, \sin\psi_b, 0] \\ \vec{r}_b &= r_b[\cos\psi_b, \sin\psi_b\cos\delta, \sin\psi_b\sin\delta] \end{aligned} \tag{4}$$

With a few steps of algebra, it can be shown that

$$|\vec{r}_a - \vec{r}_b|^2 = r_a^2 + r_b^2 - 2r_a r_b[\cos\psi_a \cos\psi_b + \cos\delta \sin\psi_a \sin\psi_b]$$

This equation can be rewritten utilizing the transformation $\psi_1 = \psi_a + \psi_b$ and $\psi_2 = \psi_a - \psi_b$

$$|\vec{r}_a - \vec{r}_b|^2 = r_a^2 + r_b^2 - r_a r_b[\cos\psi_1(1-\cos\delta) + \cos\psi_2(1+\cos\delta)] \tag{5}$$

We will refer to ψ_a as the antisynodic phase and ψ_2 as the synodic phase. This is the fundamental concept that differentiates this paper from [4]. Instead of defining relative motion using spherical geometry, here, it is being defined from a pseudo-signal processing perspective. Next we take the derivatives of the relative distance squared from equation 5, which will be relevant to the relative velocity and acceleration.

$$\begin{aligned}\frac{d|\vec{r}_a - \vec{r}_b|^2}{dt} = 2r_a\dot{r}_a + 2r_a\dot{r}_a \\ - (\dot{r}_a r_b + r_a \dot{r}_b)[\cos\psi_1(1-\cos\delta) + \cos\psi_2(1+\cos\delta)] \\ + r_a r_b[\sin\psi_1 \dot{\psi}_1(1-\cos\delta) + \sin\psi_2 \dot{\psi}_2(1+\cos\delta)]\end{aligned} \tag{6}$$

and

$$\begin{aligned}\frac{d^2|\vec{r}_a - \vec{r}_b|^2}{dt^2} = 2(\dot{r}_a^2 + r_a\ddot{r}_a) + 2(\dot{r}_b^2 + r_b\ddot{r}_b) \\ - (2\dot{r}_a\dot{r}_b + r_a\ddot{r}_b + r_b\ddot{r}_a)[\cos\psi_1(1-\cos\delta) + \cos\psi_2(1+\cos\delta)] \\ + 2(r_a\dot{r}_b + \dot{r}_a r_b)[\sin\psi_1 \dot{\psi}_1(1-\cos\delta) + \sin\psi_2 \dot{\psi}_2(1+\cos\delta)] \\ + (r_a r_b)[\cos\psi_1 \dot{\psi}_1^2(1-\cos\delta) + \cos\psi_2 \dot{\psi}_2^2(1+\cos\delta)] \\ + (r_a r_b)[\sin\psi_1 \ddot{\psi}_1(1-\cos\delta) + \sin\psi_2 \ddot{\psi}_2(1+\cos\delta)]\end{aligned} \tag{7}$$

With these derivatives, we can end this subsection with the following relative equations of motion.

$$rel\_dist = \sqrt{|\vec{r}_a - \vec{r}_b|^2} \tag{8}$$

$$rel\_vel = \frac{\frac{d|\vec{r}_a - \vec{r}_b|^2}{dt}}{2rel\_dist} \tag{9}$$

$$rel\_accel = \frac{\frac{d^2|\vec{r}_a - \vec{r}_b|^2}{dt^2}}{2rel\_dist} - \frac{rel\_vel^2}{rel\_dist} \tag{10}$$

2.3 The Relative Distance Equation

The final equation for relative distance between two arbitrary satellites at any given time is repeated below from clarity.

$$|\vec{r}_a - \vec{r}_b| = \sqrt{r_a^2 + r_b^2 - r_a r_b[\cos\psi_1(1-\cos\delta) + \cos\psi_2(1+\cos\delta)]}$$

Under two-body motion, δ is a constant and for near-circular orbits, r_a and r_b are nearly constant, leaving only ψ_1 and ψ_2 as time-dependent variables. The following figure is an example of the relative distance between two satellites with low eccentricities and small relative inclination.

It is clear that there are two signals at play. The signal with the longer period is the synodic (ψ_2) motion , while the shorter period is the antisynodic (ψ_1) motion. The synodic motion will be utilized for describing a new prefilter, while the antisynodic motion can be utilized to find local minimums.

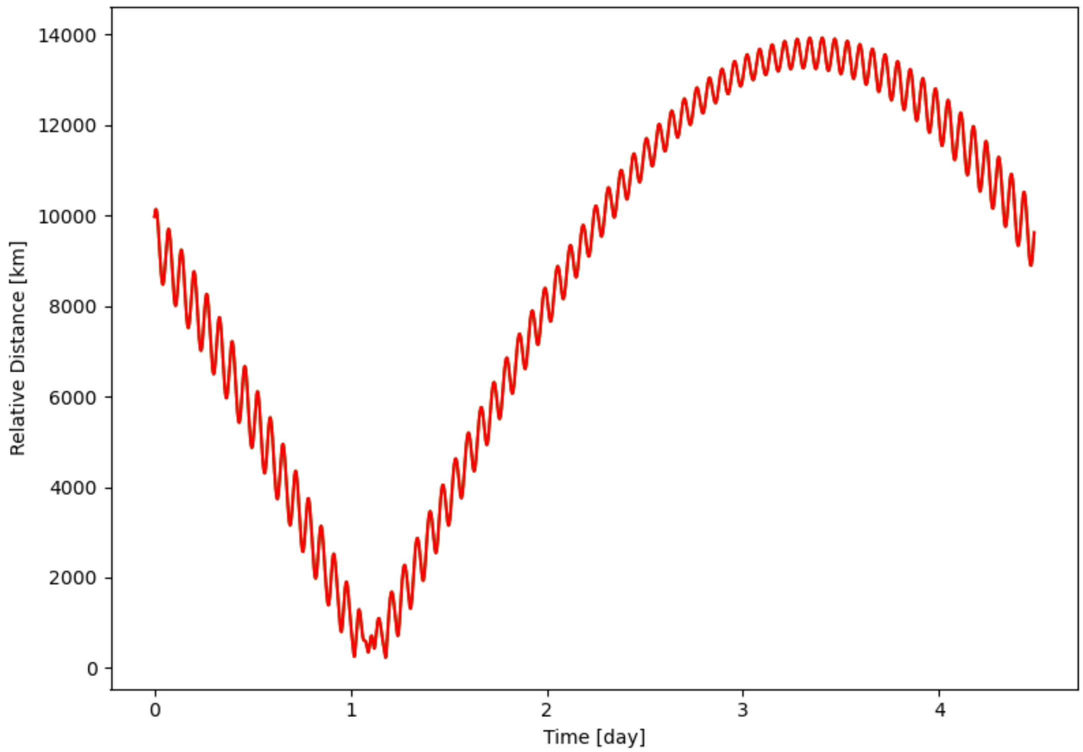

Fig. 1: Relative distance between two orbits with orbital elements (a, e, i, Ω, ω, θ):
6769.09, 0.02645, 0.00, 0.00, -72.56, -83.56
6834.42, 0.06764, 4.00, 0.00, -109.90, 51.17

It should be noted that the relative weights of $(1+\cos\delta)$ for ψ_2 and $(1-\cos\delta)$ for ψ_1 are dependent upon δ. As the relative inclination increases, the anti-synodic motion becomes more dominant. While it is not exactly accurate to equate anti-synodic motion with out-of-plane motion, it is a helpful description. When two objects are coplanar, or δ is equal to 0, this term vanishes completely. As objects approach $\delta = 90$ degrees, the two oscillations carry the same weight making the above pattern harder to visualize.

2.4 Time Prefilters

While the geometric prefilters mentioned in the introduction act as a Boolean test whether or not we should consider a given pair of satellites, the time prefilter reduces the time where we must investigate if two satellites are within a threshold. In [4], Hoots et all introduces a single time prefilter which was an accurate description in 1984, however, this paper proposes a second time prefilter requiring a new naming convention. The original time prefilter will be referred to as the cross-track prefilter while the new time prefilter will be referred to as the synodic prefilter.

2.4.1 Cross-Track Prefilter

The cross-track prefilter bounds the potential range of true anomalies for both satellites by returning windows where the cross-track distance of a given satellite with respect to its partner's plane is less than the provided threshold. The simplest way to imagine this in a geometric sense is determining when a seconday satellite passes into the region bound by planes parallel to the primary orbital plane one $\pm$ threshold away. The next step is to calculate when the secondary satellite exists within this boundary. Lastly, switch satellies and repeat comparing these sets of time windows to each other.

Using the new mutual frame, this prefilter can be significantly easier to formulate. As we know from equation 4, the z-position of the secondary satellite can be written as

$$\vec{r}_{b_z} = r_b \sin\psi_b \sin\delta$$

Setting the r.h.s of the above equation to be less than a threshold and rearranging for ψ_b, we see

$$\sin\psi_b < \frac{thresh}{r_b \sin\delta}$$

In order to account for the eccentric nature of orbit b, we are left with the following equality that will work for all times under two-body motion,

$$\sin\psi_b < \frac{thresh}{a_b(1-e_b)\sin\delta} \tag{11}$$

The next steps are to repeat this calculation for the other obit, convert bounds on ψ to time windows, and to see where the time windows overlap.

This prefilter is particularly effective at reducing the computation load of comparing pairs of satellites with large relative inclination. In fact, as $\delta = 0$ we see that the denominator goes to 0. In the coplanar case, such as mega constellations, this prefilter provides no benefit. However, for the vast majority of comparisons between two random satellites, this prefilter does a lot of heavy lifting.

2.4.2 Synodic Prefilter

The synodic period is the amount of time it takes two bodies under two-body motion to return to the same relative positions. This term is often only used with regard to interplanetary travel and is the reason why missions to Mars only take place every 2 years or so. This prefilter utilizes that concept on the scale of Earth-orbiting satellites. Much like the cross-track prefilter, this prefilter seeks to put bounds on ψ_2, which can then be converted into time windows, which then can be compared to the valid time windows from the cross-track prefilter.

Referring back to equation 5,

$$|\vec{r}_a - \vec{r}_b|^2 = r_a^2 + r_b^2 - r_a r_b[\cos\psi_1(1-\cos\delta) + \cos\psi_2(1+\cos\delta)]$$

Next, we take the r.h.s of the equation above and require it to be less than a given threshold and isolate for ψ_2. Notably, we set $\cos\psi_1 = 1$.

$$\cos\psi_2 > \frac{r_a^2 + r_b^2 - thresh^2}{2r_a r_b(1+\cos\delta)} - \frac{1-\cos\delta}{1+\cos\delta}$$

Lastly, the eccentricities of both orbits must be taken into account

$$\cos\psi_2 > \frac{(a_a(1-e_a))^2 + (a_b(1-e_b))^2 - thresh^2}{2a_a a_b(1-e_a)(1-e_b)(1+\cos\delta)} - \frac{1-\cos\delta}{1+\cos\delta} \tag{12}$$

A helpful way to conceptualize this prefilter is as a law of cosines-like restriction on two position vectors. If we simplify the problem and consider the coplanar case where $\delta = 0$, we can see that equation 5 reduces to the law of cosines.

$$|\vec{r}_a - \vec{r}_b|^2 = r_a^2 + r_b^2 - 2r_a r_b \cos\psi_2$$

As mentioned before, this may give the impression that ψ_1 corresponds to the out-of-plane oscillation since in the coplanar case it does not exist. It is merely out-of-plane like.

3. METHODOLOGY

The aim of this paper was to evaluate how effective the synodic prefilter was at reducing the amount of time required to propagate satellites and evaluate for close approaches. This was achieved by evaluating the synodic prefilter in the appropriate context of a full conjunction assessment algorithm. The figure below outlines the implemented algorithm, which includes both geometric prefilters: perigee-apogee test and MOID test, as well as both time prefilters.

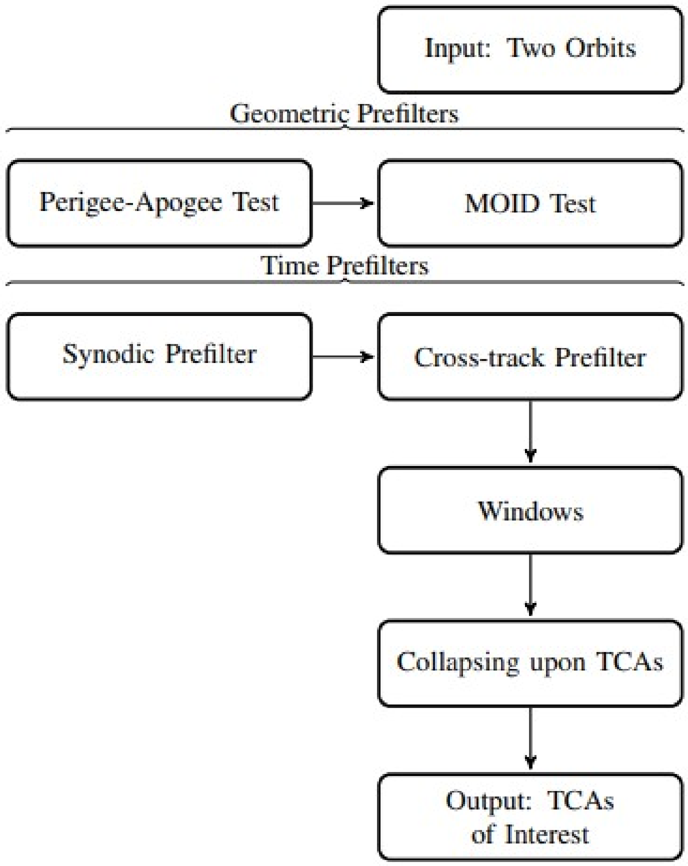

Fig. 2: Implemented conjunction assessment algorithm

6200 TLEs were pulled from Celestrack.com using the ”active” group. Their mean elements were taken and turned into a ”state” that was propagated using exclusively two-body motion. While these ”states” do not represent actual satellite positions and velocities, they do occupy the neighborhood of actual satellites and seemed like an realistic test case. I did not have access to higher fidelity statevector data.

If a given pair passed both geometric prefilters, then it was sent to the time prefilters. Windows that needed to be investigated using propagation were generated by the two time prefilters. The amount of total days in the cross-track prefilter windows was compared to the number of days in the overlapping windows from the synodic and cross-track prefilters together.

4. RESULTS

The synodic prefilter shines when the pair of satellites being compared have a large synodic period or have semi-major axes very close to one another. This can also be visualized as having similar average angular momentums so that one does not get lapped by its partner often. For example, in the case of mega-constellations with 60 or so satellites in the same plane with the same size orbits, we would need to investigate 1830 for the full time of consideration versus just

Table 1: Results comparing the effectiveness of adding the Synodic Prefilter. Across the three sample dates, there was an average improvement factor of 2.212.

Run	**Initial** [comparisons]	**Post-geo filters** [comparisons]	**Post cross-track filter** [days]	**Post both time prefilters** [days]	**Reduction Factor**
1	1.922e7	5312	.01670	.00426	3.920
2	1.922e7	5583	.01223	.00727	1.692
3	1.922e7	5585	.01130	.00625	1.808
4	1.922e7	5673	.00841	.00503	1.672
5	1.922e7	5862	.00995	.00505	1.970

60. 1830 being found by taking the triangular sum of 60 and 60 by comparing each satellites to the one to it's side until the loop is complete.

As the industry moves away from large super-redundant satellites to constellations of smallsats, the synodic prefilter will prove more and more useful. Here we can already see the amount of time needed to propagate and evaluate for TCAs halved.

5. FURTHER WORK

As previously discussed, the antisynodic motion can be utilized to nearly-analytically find the actual TCAs. In the case of nearly coplanar orbits, the synodic phase (ψ_2) must be near 0 for a close approach to occur. Solving for when the antisynodic phase (ψ_1) also equals 0 provides the exact TCA. However, as the relative inclination increases, more weight is thrown behind the antisynodic phase and the TCA becomes harder to analytically solve. This is compounded when the orbits are considerably elliptical. Checking at a rough guess of when $\psi_1 = 0$ for every antisynodic period and then using Newton's method to find the actual close approach was a reliable way to find TCAs under two-body motion. This quickly fell apart when J2 was added to the propagation. As a result, I believe there is a lot of potential to leverage this new parameter.

Additionally, all of the prefilters as described above are not robust enough to take into account perturbations. Hoots does address this in [4], proposing modifications to both geometric prefilters as well as the cross-track prefilter. I have yet to work through the analog for the time filters using this approach.

Laslty, for nearly coplanar, low eccentricity pairs of orbits, the MOID occurs at $\psi_1 = 0$ and $\psi_2 = 0$. There is potential that this new frame could provide a quick MOID algorithm, although I am not sure there is much need for that. As δ and/or $e_{a,b}$ increase, the MOID moves around in the $\psi_{1,2}$ space.

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