Metric Tensor Fields along Trajectory Solution Surfaces for Astrographic Map-Making

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ABSTRACT

Astrography, an extension of geography to space domains, aims to communicate information connecting human and mission factors to the spatial dimensions. Part of the difficulty in communicating geographic factors to decision makers is the necessity to often display the information on a table-top map, a dimension reduction from the manifold that fully describes the space. With this dimension reduction comes a loss of information, and the projection distorts spatial quantities. This becomes extremely important when creating astrographic maps due to inclusion of velocity and time in addition to position in the state space. In astrography, spatial distance may be less important than other distance metric quantities, thus visualizing additional mission objectives in the form of distance metrics becomes a priority in creating these maps. In addition, velocity and time are difficult to visualize alongside position, requiring a form of metric coordinatization or coordinate patches to communicate the mission space astrographically.

1. INTRODUCTION

Visualizations are used for decision making in various fields; commerce and strategy on land, sea, and air particularly benefit from specialized maps. Visual tools like maps are useful for decision makers when the decisions are analytically dependent on the space [10]. Ships navigating during Europe's overseas explorations benefitted from maps such as the Mercator projection that translates a ship's bearing to direction on the Earth using the projection map's preservation of angles [7, 4]. Meanwhile, National Geographic currently uses Winkel-Tripel maps in general because it compromises amounts of distortion of the common properties including angles and areas, for the purpose presenting knowledge for the general public [4]. These visualizations are useful when a globe (that doesn't distort any properties) either isn't practical or a lower-dimensional table-top map is required. In astrogrpahy, there exists no direct analog to a globe because at minimum the space domain exists in $\mathbb{R}^6$ space. Additionally, the projection of the higher-dimensional space can include multiple important factors to increase the information contained within a lower-dimensional map. This has precedent in projection maps that show population density over land and traffic densities over sea as in figure 1. A metric tensor provides the mathematical measurement of the distortion to incorporate higher-dimensional quantities onto a map projected from the original space. This tensor is derived from the definition of the distance metric of interest (according to the mission objectives), as well as the coordinates that are relevant to the mission. Metric tensors also provide the method of determining and describing the stretching of the map, as can be seen in Tissot indicatrices that are placed on analytical versions of projection maps [1, 4]. Indicatrices can be displayed on top of astrographic maps to show how the distances are flexed and skewed in the projection. All of these tools can be inherited from geographic mapping to make effective communicative tools in astrography.

2. MANIFOLD CONSTRUCTION FROM PROBLEM

The solution space is constructed as a subset of the real-world higher-dimensional, which has dimension equal to the number of relevant states. In geographic settings, this includes almost exclusively position space, but may include other parameters depending on the factors being considered. A table-top map is generally a subspace of $\mathbb{R}^2$ space, but population density maps will include population as a relevant state, and therefore is a subspace of $\mathbb{R}^3$. In astrography,

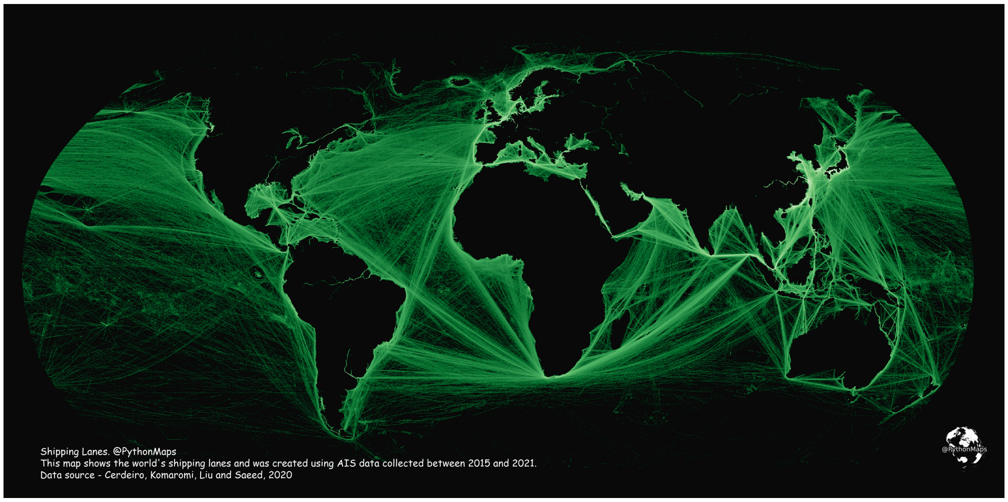

Fig. 1: Shipping lanes that highlight heaviest traffic pathways and choke points

the solution space derives from dynamics, and therefore includes both position and velocity as a subspace of $\mathbb{R}^6$. In addition, the solution space may include time in non-autonomous dynamics, as a subspace of $\mathbb{R}^7$. Any other parameters that are relevant to the decision makers in a spacecraft mission is then included in the state space dimension n.

The inclusion of time and velocity in the space makes visualization and communication significantly more difficult; position space is inherently comprehensible for map users, as the variables have been communicated in maps throughout the history of the fields of geography and cartography. An astrographic solution is unique only if its velocity is specified, and sometimes its time as well. Having a time and velocity communicated in a spatial layout such as a map is not inherently intuitive.

The primary use of the map by a user is to decide what paths to take; this is traditionally done by considering optimal overall distance travelled. It is not always that this must be the shortest distance travelled, as effort can be an important factor as well, and a short distance over land can traverse mountains that require considerable effort. The effort expended, as a control parameter for the traveller, is mathematically an independent factor in a distance as well [6]. Distance metrics need not necessarily be traditional spatial distances, as long as they follow the distance metric rules in equations 1 to 4.

$$g(\underline{x},\underline{y}) > 0 \text{ for } \underline{x} \neq \underline{y} \tag{1}$$

$$g(\underline{x},\underline{y}) = 0 \text{ iff } \underline{x} = \underline{y} \tag{2}$$

$$g(\underline{x},\underline{y}) = g(\underline{y},\underline{x}) \tag{3}$$

$$g(\underline{x},\underline{z}) \leq g(\underline{x},\underline{y}) + g(\underline{y},\underline{z}) \tag{4}$$

Constructing a map that considers the distances between solutions as quantities that are relevant to a decision maker instead of solely physical spatial distance is likely more useful. With this formulation, solutions can be compared with respect to other solutions, with transfers between solutions showing the change in that distance metric instead of spatial distance. In the case of most astrographic solutions, physical distance between solutions is not the only quantity of interest, because the position changes over time. Instead, a distance metric composed of mission objectives such as transfer time can relay more complex information like optimal otrbit transfer locations.

3. METRIC TENSOR USE IN CREATING MANIFOLD SUBSPACES

It is important for the metric of interest to comply with the formal definition of a distance metric to characterize the flexion and skewing of the $\mathbb{R}^n$ space by the distance. If the distance does not follow distance metric rules, visualization meets significant hurdles in the transition between the full space and the projected space. The use of a non-distance metric makes a metric coordinatization visualization impossible (elaborated on in Section 4). A visualization that is a projection of the state space in its original units, the distortions of the projection map can be quantified so long as the metric tensor is defined as a distance metric. This is how distortions are mathematically reflected visually using the Tissot indicatrix [1, 4]. Goldberg and Gott describe an improvement in communication of data in the Tissot indicatrix, one that better describes the flexion $\mathscr{F}$, or bending of the path from a Euclidean straight line, and the skewness $\mathscr{S}$, or lopsided stretching, of geodesics on the map, described analytically in equation 5 [4]. Using an indicatrix can lead to a greater intuition for users when there is an analytical form of the visualization, and will be especially important in early iterations of astrographic maps created.

$$\begin{gathered}\theta = \text{bearing angle} \\ \mathbf{u} = \text{local infintesimal vector of geodesic} \\ \mathscr{F} = \frac{d\theta}{dg} \\ \mathscr{S} = \frac{1}{ds}\frac{d\mathbf{u}}{dg}\cdot\mathbf{u}\end{gathered} \tag{5}$$

The creation of the distance metric from the mission objectives leads to a metric tensor in a relevant coordinate system. The metric tensor mathematically describes how the geodesics, which optimize the distance metric, curve in these selected coordinates. Indicatrices displayed on top of these projection maps then provide analytical insight into how the map curves the best paths.

4. METRIC COORDINATIZATION

Coordinates can be chosen to correspond to directions of the state space that are most relevant to the decision maker, or are relevant with respect to the distance metric. The spatial distance between points is important in most table-top geographic maps, but each projection also aims to preserve some other property or compromise between the distortions of the important properties. The distorted projection maps of the Earth may preserve some quantities, but if these quantities are distance metrics, they can be used to determine decision making coordinates, measured in the units of the distance metric. This is called metric coordinatization, in which the information contained within the coordinates are directions and distances.

Using the distance metric g, the metric coordinatizing set $\mathscr{C}$ is composed as the set of points c in the solution space $\mathscr{X}$ from which the other points x are measured by g. What results is a metric coordinatizing system $(\mathscr{M}, g, \mathscr{X}, \mathscr{C})$ in which the new representation of the points $x \in \mathscr{X}$ are defined in coordinates x_c defined in equation (6) [2].

$$\begin{gathered}x_c := g(x,c) \\ c \in \mathscr{C}, x \in \mathscr{X} \\ x_{\mathscr{C}} = (x_c)_{c\in\mathscr{C}} \in \mathbb{R}^{card(\mathscr{C})}\end{gathered} \tag{6}$$

This is shown in figure 2, where the distance metric is the great circle distance, and the first two coordinates are the distances from Washington D.C. and Cape Town, and the third coordinate is latitude. Latitude is a third metric coordinate that is the great circle distance measured up a line of longitude from the equator. Three coordinates are required for this geometry, and this does not simplify the map, but it does give a representation of how the longitude and latitude contribute to spatial distances along the surface of the Earth. Using this globe instead of the sphere in Euclidean space with Euclidean distances, a user can better intuit distances traveled from the two cities of concern.

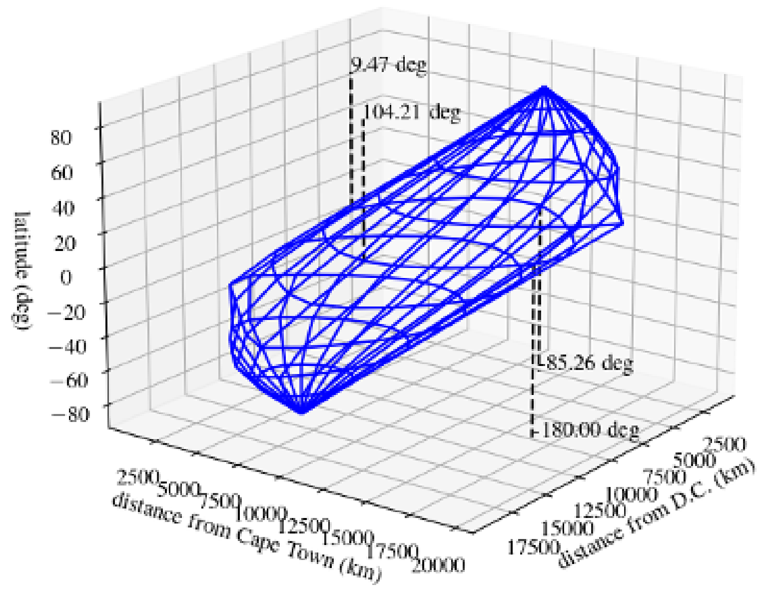

Fig. 2: A discretized metric coordinatized Earth map, with coordinates distance from DC, distance from Cape Town, and latitude

5. COORDINATE PATCHES

Metric coordinatization is not a useful tool for all purposes; the idea of presenting these distance metrics in a spatial visualization may be too confusing, and not all times are specific reference points identifiable for a problem. In these cases, the traditional mapping method is expected to be used, which is creating charts, or coordinate patches. These charts are truncations or functionally mapped subsets of the solution space, and together create an atlas that, if compiled to include all n states, can encapsulate all the data within a solution space [5]. Coordinates can be chosen initially, including with the initial solution space's nxn metric tensor, and the charts can be essentially truncations of the solution space using these coordinates such that these same coordinates are shown. The mapping function can also transform between coordinates, and the metric tensor then follows the transformation in equation 7 [9, 1].

$$\begin{aligned} G'_{ij} &= G_{kl}\frac{\partial x^i}{\partial x^k}\frac{\partial x^j}{\partial x^l} \\ ds^2 &= G'_{ij}dx^i dx^j \\ ds^2 &= G_{kl}dx^k dx^l \end{aligned} \tag{7}$$

How these charts are chosen should take from the lessons learned from GIS, particularly the usefulness of visual representations when decision makers need to make decisions connected to the space. As the state space includes the position and velocity, this is certainly the case, so maps of the space domain can be extremely useful [3]. This is where the metric tensor is helpful, when the distances are not necessarily spatial while the coordinates (positions, velocities, and time) are. There may be situations in which certain coordinates x^i play no role in the orbit, or more likely, their deltas have lesser impact on a delta in the distance metric. This manifests in the diagonal component of the matrix form of the metric tensor G_{ii}. As well, when the off-diagonal elements of the metric tensor can demonstrate effect of the coupling of deltas in two coordinates on the distance metric. Consider the distance metric computed in equation 8, which expands the arc length (the distance defined by the metric) equation 7.

$$ds^2 = G_{11}dx^1dx^1 + 2G_{12}dx^1dx^2 + 2G_{13}dx^1dx^3 + G_{22}dx^2dx^2 + 2G_{23}dx^2dx^3 + \ldots \tag{8}$$

For a case in which the change in x^1 makes little effect on the measurement of the relevant distance metric, the metric tensor element G_{11} should be small. When two coordinates are coupled, such as x^2 and x^3, their change making a significant change in the distance metric measurement should appear in a significant G_{23}. This merits then a chart using coordinates x^2 and x^3. If the same is for x^1 and x^2, then a chart visualizing coordinates x^1 and x^2 makes sense despite x^1 on its own not affecting much the arc length. This latter case is not too likely because the need of the metric tensor to be positive definite, as well as the logic of such a case existing.

In these charts, in which the coordinates reflect the spatial units, the distance metric does not necessarily get reflected in the distance between points. Topographic maps are examples of using geographic representations to demonstrate a separate metric along spatial coordinates. These will show distance metrics from points in contours, as on the Earth topographic map in figure 3 and cislunar map in figure 4, in which the distances from the moon are shown in contours of 1e5 kilometers.

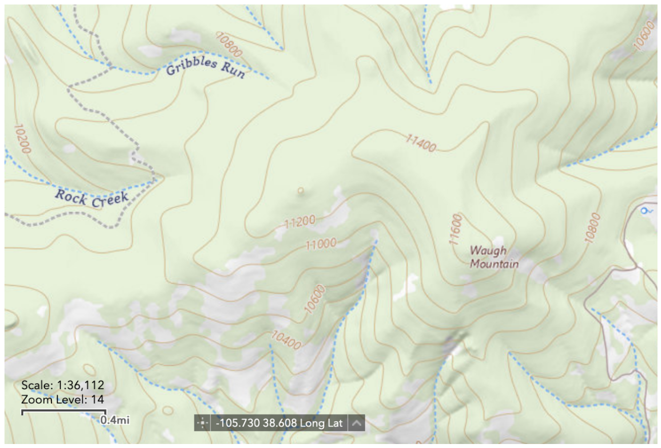

Fig. 3: A topographic map of a section of the San Isabel Forest [8]

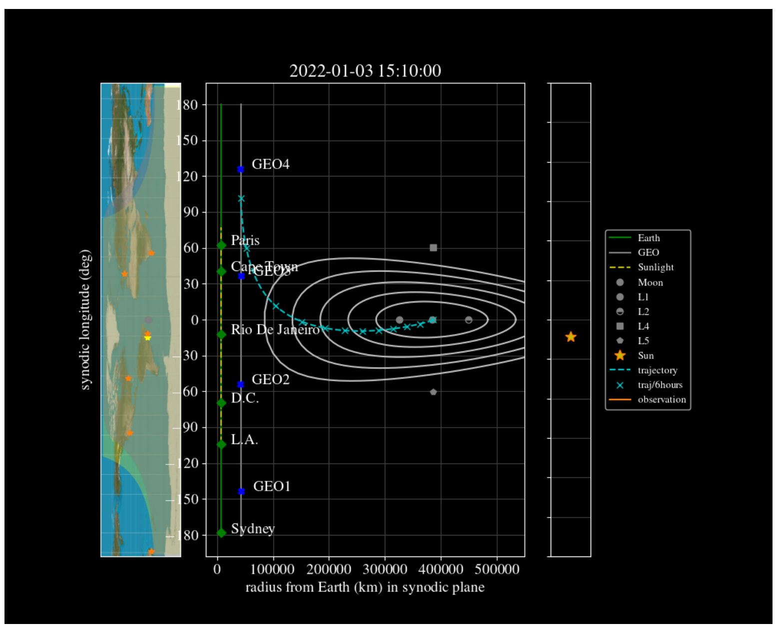

Fig. 4: A curvilinear orthographic projection including a trajectory from the moon to GEO

6. CONCLUSION

Mapping astrographic surfaces is a challenge due to the high-dimensional and non-intuitive manifolds of the space domain, and the need for likely non-spatial distance metrics to be defined. The distance metric affects the visualization in that the visual communication of the surface comes as a metric coordinatization or an atlas of charts informed by the metric tensor. These visualizations are important because maps historically, in all areas of responsibility (AOR), have been essential tools both for decision makers and common users understanding the space of the AOR. Understanding the dynamics of the space domain in a similar fashion as terrestrial geography and cartography should make space mission decision making more effective.

7. REFERENCES

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